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Mihir Bhattacharya, Ashoka University

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Mihir Bhattacharya [*]

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**Abstract**

We study *constitutional consistency* of social choice functions (s.c.f.) in a setting where voters have single-peaked preferences over a tree. This is relevant in settings where alternatives are locations spread out on a tree and a location needs to be selected for provision of a public good. An s.c.f. is *constitutionally consistent* if its outcome at any profile does not change when the profile is restricted to any subset consisting the outcome. We show that *q-threshold rules on trees* are the only s.c.f.s which satisfy *constitutional consistency*, *unanimity* and *anonymity.* These s.c.f.s specify, for each alternative, thresholds which are decreasing (increasing) on every path from a given node. These s.c.f.s then select from the range, the unique alternative which is the smallest (greatest) alternative in any restricted vote profile that receives more additive votes than the threshold assigned to it. These s.c.f.s are generalizations of *min*, *max*, and *median* s.c.f.s when restricted to paths.

**JEL classification:** D70, D71

**Keywords:** Constitutional consistency, social choice functions, single-peaked preferences, trees.

[*]Department of Economics, Ashoka University, Rajiv Gandhi Education City, Rai, Sonipat, Haryana-131029, INDIA. Email: mihir.bhattacharya@ashoka.edu.in
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# 1 Introduction

In many voting situations, candidates or alternatives may drop out or become unavailable due to certain unforeseen circumstances. In such cases, the aggregation rule must specify what the outcome would be for every such contingency. In this paper we focus on a consistency condition which is crucial under such conditions. *Constitutional consistency* states that the outcome of a social choice function (s.c.f.) at any preference profile should be the same when *restricted* to any subset containing that outcome. This can also be interpreted as the social choice variant of Sen (1969) and Chernoff (1954)'s *contraction consistency* or $\alpha$ condition defined for individual choice functions. In this paper we study *constitutionally consistent* social choice functions when the preferences are single-peaked over a tree.[1]

Alternatives are often distributed over a tree in many settings. For example,

(1) When alternatives are locations in a city with many 'branches' and a public good needs to be provided at one of these locations.

(2) Alternatives may be researchers or professionals connected by relevance (like a tree) and one of them is to be selected for an award.

Single-peakedness is a natural assumption to make in these settings. It requires that individuals have a 'peak' or a most preferred policy and that alternatives closer to the peak are strictly preferred over the ones further away. In the first example, the individuals would always prefer to have the allocated good at a location closest to their own location and in the second example, individuals would prefer that an individual closest to their area of interest be awarded. Single-peaked preferences on a line were first introduced by Black et al. (1958), Hotelling (1929) and Downs (1957) and continue to be used in political economy as well as social choice settings.[2]

There are a few works which study this type of consistency of social choice functions in the unrestricted domain. Blair et al. (1976) provide different impossibility theorems using *Chernoff's conditions aka contraction consistency* along with other axioms like *independence of irrelevant alternatives* and *path independence*. Bandyopadhyay (1984) finds that instead of contraction con-

[1] A formal definition is provided in the model section. Intuitively, a preference is single-peaked over a tree, if (i) there exists a peak of the preference $x$ which is strictly preferred over every other alternative and (ii) alternatives further away from the peak in any 'direction' are strictly worse-off.

[2] See Arrow et al. (2010) and Austen-Smith and Banks (2005) for further exposition on social choice models with single-peaked preferences.

sistency it is expansion consistency which is problematic for the existence of non-dictatorial social choice rules.

An axiom that is very similar to ours is *self-selectivity* as introduced by Koray (2000) and further extended to single-peaked domains by Bhattacharya (2019).[3] A social choice function is *self-selective* if it chooses itself when subjected to a 'vote' against other voting rules. The latter notion and our notion seem different at the outset but they are essentially the same. Both the notions are 'consequentialist' in the sense that they only care about the alternative being chosen. Therefore, when the set of available voting rules changes, it is equivalent to a contraction of the set of alternatives over which the profile is constructed. Lainé et al. (2016) study a similar notion of stability for scoring rules and social welfare functions, and show that there are no *hyper-stable* scoring rules. Barberà and Jackson (2004) study a version of self-stability of voting rules over two alternatives, and find that majority rules are the only self-stable rules.

We characterize *q-threshold rules on trees* which are defined as follows. Pick any terminal node or alternative $r$ on the tree. Assign monotone decreasing (or increasing) thresholds to each alternative on every path $[r, y]$ for any other terminal node $y$. These rules then pick the unique alternative $x^*$ which when restricted to any path consisting of $x^*$ is such that it is the smallest (greatest) alternative from the *range* of the profile that receives more cumulative votes at the top (votes received by all the alternatives which are closer to (or further away from) $r$) than the threshold assigned to them.[4] The thresholds are defined for each alternative and depend on the path they are defined over.

Another condition that needs to be satisfied for the above rule to be consistent with itself is as follows. Suppose $x$ is an alternative which has a degree greater than or equal to 3, and $q_x^{[r,y']}$ and $q_x^{[r,y'']}$ are the two thresholds of $x$ in two extremal paths $[r, y']$ and $[r, y'']$. The condition requires that $q_x^{[r,y']} + q_x^{[r,y'']} < n + 2$. This condition ensures that the rule is well-defined.

*q-threshold rules* can be seen as generalized *quota rules* defined for trees but with variable thresholds which apply to cumulative votes for alternatives at the top. These rules can be defined with respect to any terminal node $r$ and by assigning decreasing or increasing thresholds to alternatives on every

[3] Koray and Unel (2003) extends the analysis to the *tops-only* domain and Koray and Slinko (2008) characterizes self-selective rules when the inefficient ones are excluded.

[4] The *range* of the profile is the set containing alternatives which lie on the path between at least one pair of alternatives at the top of the profile.

extremal path from $r$.[5]

We prove our main result in steps. We first show that if a social choice function is *constitutionally consistent* and *unanimous* then it is *tops-only* (Proposition 1). *Constitutional consistency* then implies that any *constitutionally consistent* and *unanimous* social choice function will only choose from the range of the profile. Finally, we use the results in Bhattacharya (2019) and the fact that the two versions of *constitutional consistency* are the same under the tops-only property. The main proof overcomes significant challenges since the rules must be consistent across different extremal paths in the tree. Once that is resolved, we show that *q-threshold rules over trees* are the only *constitutionally consistent*, *anonymous* and *unanimous* social choice functions over trees.

There are some differences between our notion of consistency and the ones in the literature. First, our definition of *constitutional consistency* is applied on restriction of preference profiles, while the literature mostly defines self-selection using an induced profile over different aggregation rules and then use neutrality to check for the self-selectivity. Second, the standard notion of contraction consistency applied in the classical social choice literature (Bandyopadhyay (1984), Blair et al. (1976)) apply the condition to the social decision function without describing the preferences of individuals over the contracted set. In our notion, we implicitly assume that individuals are truthful and do not change their preferences over the feasible set of alternatives.

An important contribution of the paper is that it provides a characterization of a large class of social choice functions in a relatively general domain which has shown promise for existence results in the literature. This fact that the domain is partially ordered helps in obtaining such results. Further research on generalized single-peaked domains is required while studying similar properties of social choice functions.

The paper is organised as follows. In Section 2 we introduce the notation and describe the model. Section 3 lists the Axioms and Section 4 provides the results. This is followed by some concluding remarks. The proofs of all the results are provided in the Appendix. The bibliography is provided at the end.

[5] A path $[a, b]$ is *extremal* if both $a$ and $b$ are terminal nodes.

## 2 The Model

In this section we describe the model and provide all the definitions. There is a finite set of voters $N = \{1, 2, \ldots, n\}$ and a finite set of alternatives $X$ with $|X| = m$. Let $\mathcal{P}(X)$ denote the set of all non-empty subsets of $X$. We assume that all the alternatives are placed on a *tree*, $T \equiv T(X, E)$, where the set of nodes is the set of alternatives $X$ and $E$ is the set of edges. We will assume the tree $T(X, E)$ to be fixed for the remaining part of the paper. We will use the terms 'alternatives' and 'nodes' interchangeably.

An *path* $[x, y]$ from node $x$ to node $y$ in $X$, $x \neq y$ is a sequence of distinct nodes $(x_0, x_1, ..., x_k)$ such that $x_0 = x$, $x_k = y$ and $\{x_q, x_{q+1}\} \in E$ for all $q \in \{0, 1, ..., k-1\}$. For simplicity of notation, we also denote $[x, y]$ as the set of alternatives (including $x$ and $y$) in the path $[x, y]$ for any distinct $x, y \in X$. A path $[a, b]$ is *extremal* if both $a$ and $b$ are end nodes. Let $\mathbf{E}$ denote the set of all extremal paths in $T$. An alternative $x$ is *adjacent* to $x^+$ if $[x, x^+] = \{x, x^+\}$.

**Single-peaked preferences over trees:** Each voter $i \in N$ has a single-peaked preference on $T$ which is defined as follows. A strict preference ordering, $\succ_i$, is *single-peaked on the tree* $T$ if there exists a 'peak' $x_i \in X$ such that for all $x, y \in X, x \neq y$,[6]

$$x \in [x_i, y] \Rightarrow x \succ_i y.$$

Note that in the above definition if $x = x_i$ then for any $y \neq x_i$ we have $x_i \succ_i y$. This definition reduces to the standard definition of single-peaked preferences if the tree $T$ is a 'line'.

Therefore, alternatives closer to the peaks are strictly preferred over the alternatives further away. However, the definition does not impose any restriction on two alternatives which are on either side of the peak or, in other words, when neither is in the path between the peak and the other alternative. For example, in Figure 1 voter $i$ with peak $a$ can have either $b \succ_i d$ or $d \succ_i b$.

We consider the whole set of strict single-peaked preference orderings on trees. For example, suppose $X = \{a, b, c, d, e\}$ and the tree is $T$ as shown in Figure 1. Then, the set of all strict single-peaked preferences on the given

[6]A strict binary relation $\succ$ is an ordering if it is: (i) Complete: For all $x, y \in X$, $x \neq y$ either $x \succ y$ or $y \succ x$ (ii) Transitive: If for all $x, y, z \in X$ $x \succ y$ and $y \succ z$ implies $x \succ z$, and (iii) Irreflexive: $\neg[x \succ x]$ for all $x \in X$.

tree with peak $a$ include the strict preferences $acbde, acdbe$, and $acdeb$ where alternatives are in the decreasing order of strict preference.[7]

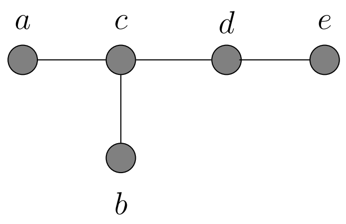


Figure 1: Single-peaked preference on $T$

Let $\mathcal{S}(T)$ be the set of all single-peaked preference orderings on $T$. A *preference profile* $\pi = (\succ_1, \ldots, \succ_n) \in \mathcal{S}(T)^n$ is a collection of $n$ preferences with peaks or 'tops', $\tau(\pi) = (\tau_1(\pi), ..., \tau_n(\pi))$. Let $\tau_i(\pi)$ denote the top of voter $i$ in the profile $\pi$. We denote by $\pi_S$ as the restriction of the profile $\pi$ to a subset $S \in \mathcal{P}(X)$. For any $S \in \mathcal{P}(X)$ let $\mathcal{S}(T_S)^n$ be the set of all profiles which are restrictions of profiles in $\mathcal{S}(T)^n$ to the set $S$. We will write $\mathcal{S}(T)$ instead of $\mathcal{S}(T_X)$ for simplicity.

**Social choice function (s.c.f.):** A *social choice function (s.c.f.)* is a mapping $f : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ such that for any profile $\pi_S \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$ we have $f(\pi_S) \in S$. Therefore, an s.c.f. operates on every restriction of a single-peaked profile on $T$ to any subset $S$ of $X$ and produces an alternative in $S$.

In this paper, we will only focus on *tops-only* s.c.f.s since our main axiom, *constitutional consistency*, along with *unanimity* will imply this property. We provide some examples of such s.c.f.s in this setting.

**Dictatorial s.c.f.** An s.c.f. $f^i : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$, for a given voter $i \in N$ is dictatorial if $f^i(\pi_S) = \tau_i(\pi_S)$ for all $\pi_S \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$. We introduce some definitions for our next rule.

*Range of a profile:* For any profile $\pi_S \in \mathcal{S}(T_S)^n$, the *range* of the profile is the set of all alternatives that lie on the path between a pair of top-ranked alternatives in $\tau(\pi_S)$, i.e.,

$$\mathit{Range}(\pi_S) = \{x \in S : x \in [\tau_i(\pi_S), \tau_j(\pi_S)] \text{ for some } i, j \in N\}.$$

[7]Single-peaked preferences on trees defined in Schummer and Vohra (2002) are based on a notion of 'distance' and, therefore, the preferences in their model are uniquely identified by the 'peaks' of the individuals.

In Figure 1, for a three-voter profile $\pi \in \mathcal{S}(T)^3$ with $\tau(\pi) = (a, b, d)$, $Range(\pi) = \{a, b, c, d\}$. We introduce some notation to define our next rule which is defined on extremal paths.

*Thresholds on* $[r, y]$*:* For an extremal path $[r, y] \in \mathbf{E}$, we define thresholds $q^{[r,y]} : X \to N$ which will be *monotone decreasing* on $[r, y]$, i.e., $[x \in [r, x']] \Rightarrow [q_x^{[r,y]} \geq q_{x'}^{[r,y]}]$ for all $x, x' \in [r, y]$. In other words, the thresholds are said to be monotone decreasing with respect to $[r, y]$ if alternatives further away from $r$ have a weakly lower threshold.

For $r \in X$, and for any given $y \in X$ such that $[r, y] \in \mathbf{E}$ we define a complete strict ordering $<_r$ on $[r, y]$: $x <_r x'$ if and only if $x \in [r, x']$ for all distinct $x, x' \in [r, y]$. For any $x, x' \in [r, y]$ we say that $x \leq_r x'$ if either $x = x'$ or $x <_r x'$. Therefore, $x <_r x'$ if and only if $x$ is strictly 'closer' to $r$ than $x'$. Let $n_x$ the number of voters who have $x$ at their top, and let $\mathcal{S}(T_{[r,y]})$ denote the set of all single-peaked preferences defined over $[r, y]$ according to $\leq_r$. Note that if $\pi \in \mathcal{S}(T)$ is a profile of single-peaked preferences defined over $X$, then $\pi_{[r,y]}$ is a profile of single-peaked preferences defined over $[r, y]$ according to the order $\leq_r$.

**q-threshold rule on a path** $[r, y]$**:** An s.c.f. $f^q_{[r,y]} : \mathcal{S}(T_{[r,y]})^n \to [r, y]$ is a *q-threshold rule on the path* $[r, y] \in \mathbf{E}$ if there exist monotone decreasing thresholds $q^{[r,y]} : [r, y] \to N$ such that for all $\pi \in \mathcal{S}(T_{[r,y]})^n$,

$$f^q_{[r,y]}(\pi) = \underset{x \in Range(\pi)}{\arg\min} \sum_{l \leq_r x} n_l \geq q_x^{[r,y]}.$$

Therefore, $q$-threshold rules on path $[r, y]$ choose the smallest alternative in the range of $\pi$ (which is a profile in $\mathcal{S}(T_{[r,y]})^n$) according to $<_r$ for which the sum of top votes received by alternatives before it according to $<_r$ is greater than the threshold assigned to it.

Alternatively, these rules can be defined with respect to monotone increasing thresholds $q^{[r,y]} : [r, y] \to N$ such that for all $\pi \in \mathcal{S}(T_{[r,y]})^n$,

$$f^q_{[r,y]}(\pi) = \underset{x \in Range(\pi)}{\arg\max} \sum_{l \geq_r x} n_l \geq q_x^{[r,y]}.$$

Bhattacharya (2019) shows that for single-peaked domain (over a line) these rules are the only voting rules which satisfy unanimity, anonymity and *constitutional consistency*. The following class of rules is a sub-class of $q$-threshold rules which includes *min* and *max* rules.

**Positional rules on a path** $[r, y]$**:** The above defined $q$-threshold rules can be defined as positional rules if $q_x^{[r,y]} = k$ for all $x \in [r, y]$ for a given $k \in \{1, ..., n\}$. For example, $q$-threshold rules can be defined as *min* and *max* rules for $q_x^{[r,y]} = 1$ and $q_x^{[r,y]} = n$ for all $x \in [r, y]$ respectively. They can be defined as median rules by taking $q_x^{[r,y]} = \frac{n}{2}$ if $n$ is even, and $q_x^{[r,y]} = \frac{n+1}{2}$ (which picks the left-median) if $n$ is odd for all $x \in [r, y]$. We now provide the axioms.

# 3 Axioms

**Constitutional consistency.** An s.c.f. $f$ is *constitutionally consistent* if for all $\pi \in \mathcal{S}(T)^n$ and for any $S' \in \mathcal{P}(X)$,

$$[f(\pi) \in S'] \implies [f(\pi) = f(\pi_{S'})].$$

*Constitutional consistency* requires that the s.c.f produce the same outcome at $\pi$ as the one it produces at any restriction of the profile to any subset $S'$ containing $f(\pi)$. This axiom is a version of Sen (1977)'s 'contraction consistency' applied to social choice functions.

It is easy to check the dictator rule $f^i$ is *constitutionally consistent.* Consider the following arguments: for any profile $\pi \in \mathcal{S}(T)^n$, we have $\tau_i(\pi) = \tau_i(\pi_S)$ for all $S \in \mathcal{P}(X)$ if $\tau_i(\pi) \in S$. Therefore, $f^i(\pi) = \tau_i(\pi) = f^i(\pi_S) = \tau_i(\pi_S)$. We require some standard axioms in addition to the above axiom for our main result.

**Anonymity.** An s.c.f. $f$ satisfies *anonymity* if for all bijections $\sigma : N \to N$ and for all $\pi \in \mathcal{S}(T_S)^n$,

$$f(\pi) = f(\pi_\sigma)$$

where $\pi^\sigma = (\pi_{\sigma(1)}, ..., \pi_{\sigma(n)})$ is the profile of permuted preferences. Anonymity states that permuting the preferences of voters does not change the outcome.

**Unanimity.** An s.c.f. $f$ satisfies *unanimity* if for all $\pi \in \mathcal{S}(T_S)^n$ such that $\tau_i(\pi) = a$ for all $i \in N$, then

$$f(\pi) = a.$$

*Unanimity* requires that when every voter has the same peak then the outcome must be the peak. We define the *tops-only* property below.

**Tops-only.** An s.c.f. $f$ satisfies *tops-only* if for all $\pi, \pi' \in \mathcal{S}(T_S)^n$ such that $\tau(\pi) = \tau(\pi')$,

$$f(\pi) = f(\pi').$$

Our first result shows that any s.c.f. that is *constitutionally consistent* and unanimous must be tops-only.

# 4 Results

**Proposition 1** *If an s.c.f.* $f : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ *is constitutionally consistent and unanimous then it is tops-only.*

**Proof.** See Appendix.

Therefore, the outcome of an s.c.f. which is *consistent* and *unanimous* depends only on the peaks of voters. An implication of this is that $f(\pi) \in Range(\pi)$ for all $\pi \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$. In the next Proposition, we apply the results of Bhattacharya (2019) to our setting.

**Proposition 2** *Consider any* $[a, b] \in \mathbf{E}$*. Let* $f_{[a,b]}$ *be an s.c.f. on* $[a, b]$ *which is constitutionally consistent, unanimous and anonymous. Then it is a* $q$*-threshold rule on* $[a, b]$*.*

By Proposition 1 we know that $f_{[a,b]}$ will be a tops-only rule. This implies that we can apply the results of Bhattacharya (2019) to this s.c.f. and it must be a $q$-threshold rule with monotone decreasing thresholds $q_x^{[a,b]}$ on the path $[a, b]$ for any $x \in X$. Note that the following ordering over the set of alternatives $[a, b]$ can be used: $x \leq_a y$ if and only if $x \in [a, y]$. When extending the rule to multiple paths, we only need to define $q$-threshold rules from a given terminal node (say $r$).

Next we define a $q$-threshold rule on $T$. Let $f|_S$ for any $S \subseteq X$ be the restriction of $f$ to the set $S$.[8] For any terminal node, $r \in X$, let $\mathbf{E}_r$ denote the set of all extremal paths with $r$ as an end-point. Let $deg(x)$ denote the degree of the alternative (or node) $x \in X$ in $T$.

[8]A function $f_D$ defined on a set $D \subseteq X$ is said to be restriction of $f$ (defined on $X$) to $D$ if $f(x) = f_D(x)$ for all $x \in D$.

**Definition 1 (q-threshold rule on $T$)** *An s.c.f. $f_r^q : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ is a q-threshold rule on $T$ with respect to $r \in X$ if there exists a strict single-peaked ordering $<_r$ on $T$ and thresholds $q : X \times \mathbf{E}_r \to N$ which are (i) monotone decreasing on any extremal path $[r, y] \in \mathbf{E}_r$ and (ii) for all $x \in T$ such that $deg(x) \geq 3$, $q_x^{[r,y]} + q_x^{[r,y']} < n + 2$ for all distinct paths $[r, y], [r, y'] \in \mathbf{E}_r$ such that $x \in [r, y] \cap [r, y']$. Moreover, for all $\pi \in \mathcal{S}(T)^n$,*

$$f_r^q(\pi) = x^* = \underset{x \in Range(\pi_{[r,y]})}{\arg\min} \sum_{l \leq_r x} n_l \geq q_x^{[r,y]},$$

*for all $[r, y] \in \mathbf{E}_r$ such that $x \in [r, y]$.*

Therefore, an s.c.f. is a $q$-threshold rule on $T(E)$ for a given $r \in X$ satisfies the following properties: (i) the restriction of the rule to any extremal path $[r, y]$ in $\mathbf{E}$ is a $q$-threshold rule on $[r, y]$, (ii) for any alternative which has degree greater than or equal to 3, the sum of its thresholds on two distinct extremal paths must be strictly less than $n + 2$ and (iii) the outcome at any profile is the unique alternative, $x^*$, which is the outcome of the rule over restrictions of the profile to any extremal path $[r, y]$ which consists of $x^*$. Alternatively, these rules can also be defined using monotone increasing thresholds on extremal paths from $r$. Our main result characterizes this rule using the axioms mentioned in Section 3.

We show that this is a well-defined rule. Firstly, note that $f_r^q(\pi) \in Range(\pi)$. We first prove this for the case when $Range(\pi) \subseteq [r, y] \in \mathbf{E}$. Since $[r, y]$ is just a line, we can define $\min(\pi)$ and $\max(\pi)$ according to $<_r$. Suppose, on the contrary, that $f_r^q(\pi) \notin Range(\pi)$ and assume w.l.o.g. that $f_r^q(\pi) >_r \max(\pi)$. By the definition of arg-min in part (i) of the above definition, $f_r^q(\pi)$ cannot lie outside the range since $\max(\pi)$ is an alternative that obtains as many cumulative votes as $f(\pi)$. But since $\max(\pi) <_r f_r^q(\pi)$, the latter cannot be the smallest alternative according to $<_r$ that meets the above condition. This is a contradiction. Therefore, $f_r^q(\pi) \leq_r \max(\pi)$. Similar arguments can be made to show that $f_r^q(\pi) \geq_r \min(\pi)$. Therefore, $f_r^q(\pi) \in Range(\pi)$ if $Range(\pi) \subseteq [r, y] \in \mathbf{E}$.

We now prove the above claim more generally. Suppose for a given profile $\pi \in \mathcal{S}(T)^n$, $Range(\pi)$ is not a subset of the set of alternatives in an extremal path from $r$ and $f_r^q(\pi) \notin Range(\pi)$. Then there must exist an extremal path $[r, \bar{y}]$ such that $f_r^q(\pi) \in [r, \bar{y}]$ but $f_r^q(\pi) \notin [\min(\pi_{[r,\bar{y}]}), \max(\pi_{[r,\bar{y}]})]$. By property (iii) of the rule, the restriction of the profile to $[r, \bar{y}]$ will not change the outcome. Therefore, $f_r^q|_{[r,\bar{y}]}(\pi_{[r,\bar{y}]}) \notin [\min(\pi_{[r,\bar{y}]}), \max(\pi_{[r,\bar{y}]})]$. But this is a contradiction to the fact that $f_r^q|_{[r,\bar{y}]}$ is also a $q$-threshold rule on path

$[r, \bar{y}]$.

By the definition, the outcome of any $q$-threshold rule is the *unique* alternative (say, $x^*$) in the range of the profile $\pi \in \mathcal{S}(T)^n$ which is also the outcome of the s.c.f. $f(\pi_{[r,y]})$ for all such profiles which are restrictions of the profile $\pi$ to any extremal path $[r, y] \in \mathbf{E}_r$ given that $x \in [r, y]$. To argue this we show that such an alternative exists and is unique.

**Claim 1** *Suppose $f_r^q$ is $q$-threshold rule on $T(E)$. For every profile $\pi \in \mathcal{S}(T)^n$ there exists a unique alternative $x^* \in Range(\pi)$, such that $f_r^q(\pi) = f_r^q|_{[r,y]}(\pi_{[r,y]}) = x^*$ for all $[r, y] \in E_r$ such that $x^* \in [r, y]$.*

**Proof.** We prove by contradiction. Suppose there exist two alternatives $x'$ and $x''$ such that $x' \in [r, y']$ and $x'' \in [r, y'']$ which satisfy the condition in Claim 1. If $[r, y'] = [r, y'']$ then we arrive at a contradiction immediately due to the definition of $q$-threshold rules. Suppose $[r, y'] \neq [r, y'']$, then there exists an alternative $x \in [r, y'] \cap [r, y'']$. Pick the one furthest away from $r$; if no other alternative is available then pick $r$. Let this alternative be denoted as $\tilde{x}$. Note that $\tilde{x} \in [r, x']$ and $\tilde{x} \in [r, x'']$ by construction.

Suppose the profile is restricted to $S = \{\tilde{x}, x', x''\}$. Let $e$ be the number of peaks at $\tilde{x}$, let $c$ be the number of peaks at $x'$ and let $d$ be the number of peaks at $x''$ in the restricted profile $\pi_S$. Note that $S \subset ([r, y'] \cap [r, y''])$ which implies that when $\pi$ is restricted further to $S' = \{\tilde{x}, x'\}$ the peaks at $x''$ will be transferred to $\tilde{x}$. Similarly, when $\pi$ is further restricted to $S'' = \{\tilde{x}, x''\}$, the peaks at $x'$ will be transferred to $\tilde{x}$.

By our assumption and the definition of $q$-threshold rule on $[r, y']$ and $[r, y'']$, $f(\pi_{\{\tilde{x},x'\}}) = x'$ and $f(\pi_{\{\tilde{x},x''\}}) = x''$. The following conditions hold due to the above assumptions,

$$q_{\tilde{x}}^{[r,y']} > e + d \text{ and } q_{\tilde{x}}^{[r,y'']} > e + c \tag{1}$$

Adding the above two inequalities, we get $q_{\tilde{x}}^{[r,y']} + q_{\tilde{x}}^{[r,y'']} > 2e + d + c$. By part (iii) of the definition of $q$-threshold rule on $T$, and the fact that $e + d + c = n$, we get,

$$n + 2 > q_{\tilde{x}}^{[r,y']} + q_{\tilde{x}}^{[r,y'']} > n + e.$$

Note that the above inequalities can hold only if $e = 0$. This implies that $c + d = n$. The above observations imply that the following three expressions must hold,

$$n + 2 > q_{\tilde{x}}^{[r,y']} + q_{\tilde{x}}^{[r,y'']} > n, \;\; q_{\tilde{x}}^{[r,y']} > d, \;\; q_{\tilde{x}}^{[r,y'']} > c.$$

It is easy to verify that the above conditions cannot be met if $c+d=n$.

■

**Theorem 1** *An s.c.f.* $f : \bigcup_{S\in\mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ *is constitutionally consistent, unanimous and anonymous if and only if it is a q-threshold rule on* $T$.

**Proof.** See Appendix.

Theorem 1 provides a characterization of *constitutionally consistent* voting rules. The proof of the result relies on Proposition 1 and the fact that the restriction of these s.c.f.s are *q-threshold rules* on any path $[a, b] \in \mathbf{E}$. However, our characterization does not require the full strength of this implication. We fix a terminal node $r \in X$ and define $q$-threshold rules on any path $[r, y] \in \mathbf{E}_r$. By Proposition 1 any s.c.f which is unanimous and *constitutionally consistent* must be *tops-only*. This implies that any restriction of the rule to a path $[r, y]$ must pick an alternative in the range of the restricted profile. To ensure that restrictions of the rule to different extremal paths in $\mathbf{E}_r$ do not contradict each other, another property, called *intersectionality* is required. It states that if the outcome of a profile $\pi \in \mathcal{S}(T)^n$ when restricted to an extremal path $[r, y] \in \mathbf{E}_r$ does not lie in $[r, y] \cap [r, y']$ for another extremal path $[r, y'] \in \mathbf{E}_r$, then the outcome of $\pi$ when restricted to $[r, y']$ must lie in $[r, y] \cap [r, y']$. An implication of this property is that the sum of thresholds of an alternative which has degree greater than or equal to three on two distinct extremal paths $[r, y]$ and $[r, y']$ in $\mathbf{E}_r$ must be less than or equal to $n + 1$. This completes the characterization of the rule on $T$ using the rules defined on every extremal path $[r, y]$ in $\mathbf{E}_r$.

# 5 Conclusion

This paper characterizes the class of *constitutionally consistent* social choice functions in the single-peaked domain over trees. The s.c.f.s we characterize, *q-threshold rules on trees*, can be seen as generalized versions of positional rules such as the min, max and median s.c.f.s when restricted to a line.

# 6 Appendix

**Proof of Proposition 1** We argue that we only need to prove the claim for all $\pi \in \mathcal{S}(T)^n$. By *constitutional consistency*, $f(\pi_S)$ for any $\pi_S \in \mathcal{S}(T_S)$ for some $S \in \mathcal{P}(X)$ will be invariant to changes in the 'tops' of restricted profiles.

We first show that $f(\pi) \in Range(\pi)$ for any $\pi \in \mathcal{S}(T)^n$. Suppose for contradiction that $f(\pi) \notin Range(\pi)$. Take an alternative $x^* \in X$ which is closest to $f(\pi)$ in $T$ and also in the range of $\pi$, i.e., $x^* \in Range(\pi) \cap [\tau_k(\pi), f(\pi)]$ for some voter $k \in N$ such that there is no other $x' \in [x^*, f(\pi)] \cap Range(\pi)$. By single-peakedness over a tree, since $x^* \in [\tau_i(\pi), f(\pi)]$, we have $x^* \succ_i f(\pi)$ for all $i \in N$. By unanimity, for $S = \{x^*, f(\pi)\}$ we have $f(\pi_S) = x^*$. By *constitutional consistency*, we have $f(\pi) = f(\pi_S)$. This is a contradiction since $f(\pi) \neq x^*$. Therefore, $f(\pi) \in Range(\pi)$ for all $\pi \in S(T)^n$.


Figure 2: Proving tops-onlyness

We now prove the tops-only property. Let $\pi = (\succ_i)_{i \in N}$ and $\pi' = (\succ'_i)_{i \in N}$ such that $\tau(\pi) = \tau(\pi')$. We show that $f(\pi) = f(\pi')$. Suppose for contradiction that $f(\pi) \neq f(\pi')$. Let $f(\pi)^+$ be the alternative adjacent to $f(\pi)$ and lies in the path $[f(\pi), f(\pi')]$ (shown in Figure 2).

We construct a profile $\hat{\pi} \in \mathcal{S}(T)^n$ by changing voter preferences in $\pi$ such that,

$$\hat{\pi}_{\{f(\pi), f(\pi)^+\}} = \pi_{\{f(\pi), f(\pi)^+\}} \text{ and } \hat{\pi}_{\{f(\pi), f(\pi')\}} = \pi'_{\{f(\pi), f(\pi')\}}. \qquad (*)$$

There are three types of voters in $\pi$ and $\pi'$ whose preferences we change sequentially as follows.

Case 1: Consider a voter $i \in N$, such that $f(\pi) \in [\tau_i(\pi), f(\pi)^+]$ or $f(\pi) \in [\tau_i(\pi'), f(\pi)^+]$ (since $\tau(\pi) = \tau(\pi')$). By single-peakedness, voter $i$ prefers $f(\pi)$ to $f(\pi)^+$. Since $\tau(\pi) = \tau(\pi')$, these voters have the same top in $\pi'$ as well. Therefore, by single-peakedness, $f(\pi) \in [\tau_i(\pi'), f(\pi)^+]$ implies that $f(\pi) \succ'_i f(\pi')$. We bring $f(\pi)$ to the top of the preferences of these voters. All the alternatives $x \in [\tau_i(\pi), f(\pi)]$ can be moved below the peak but above the alternatives to the left of $x$ as we move further away from $f(\pi)$. Therefore, for any voter $i \in N$, we make the following changes:

(i) If $f(\pi) \in [\tau_i(\pi), f(\pi)^+]$ then $\tau_i(\hat{\pi}) = f(\pi)$.

(ii) For all $x, y \in X, x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$, then $x \mathrel{\hat{\succ}_i} y$.

All the other alternatives are adjusted accordingly as per the definition of single-peakedness as we move away from the peak, $\tau_i(\hat{\pi}) = f(\pi)$. This ensures that for these voters the conditions in Equation $(*)$ are met.

Case 2: Consider any voter $i \in N$ such that $\tau_i(\pi) = \tau_i(\pi') \in [f(\pi)^+, f(\pi')]$.

By single-peakedness, $f(\pi)^+ \succ_i f(\pi)$ which is consistent with the first part of $*$. We bring $f(\pi)^+$ to the top of the preference. However, their preferences may not satisfy the second condition with respect to the preference profile $\pi'$. To account for this, we make the following changes, for any voter $i \in N$,

(i) If $\tau_i(\pi) \in [f(\pi)^+, f(\pi')]$ then $\tau_i(\pi) = f(\pi)^+$. Moreover, if $f(\pi) \succ'_i f(\pi')$ then $f(\pi) \hat{\succ}_i f(\pi')$, otherwise, if $f(\pi') \succ'_i f(\pi)$ then $f(\pi') \hat{\succ}_i f(\pi)$.

(ii) For all $x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$, then $x \hat{\succ}_i y$.

Condition (i) above ensures that both parts of the Equation $(*)$ are satisfied with respect to the given alternatives, while condition (ii) ensures that the new preference, $\hat{\succ}_i$, is single-peaked with respect to all the alternatives.

Case 3: Consider any voter $i \in N$ such that $f(\pi') \in [f(\pi)^+, \tau_i(\pi)]$. All these voters will have the same preferences over the pairs $\{f(\pi), f(\pi)^+\}$ and $\{f(\pi), f(\pi')\}$ due to single-peakedness. Therefore, for these voters both the conditions in Equation $(*)$ are satisfied and no further change is required. Similar to Case 1, we change the preferences as follows:

(i) If $f(\pi') \in [\tau_i(\pi)), f(\pi)^+]$ then $\tau_i(\hat{\pi}) = f(\pi')$.

(ii) For all $x, y \in X, x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$ then $x \hat{\succ}_i y$.

Therefore, the other alternatives are adjusted accordingly as per the definition of single-peakedness as we move away from the peak, $\tau_i(\hat{\pi}) = f(\pi)$. This ensures that the conditions in Equation $(*)$ are met for these voters.

Case 4: Consider any voter $i \in N$ for which none of the above conditions are satisfied. This implies that $\tau_i(\pi) \notin [f(\pi)^+, f(\pi')]$, $f(\pi)^+ \in [f(\pi), \tau_i(\pi)]$ and $f(\pi') \notin [\tau_i(\pi), f(\pi)^+]$. In other words, these voters have peaks which lie in one of the 'branches' of the tree $T$ emanating from alternatives which lie in the path $[f(\pi)^+, f(\pi')]$ excluding $f(\pi)^+$ and $f(\pi')$. These voters satisfy the first part of Equation $(*)$ but may not satisfy the second part.

(i) If $\tau_i(\pi) \notin [f(\pi)^+, f(\pi')]$, $f(\pi)^+ \in [f(\pi), \tau_i(\pi)]$ and $f(\pi') \notin [\tau_i(\pi), f(\pi)^+]$ then let $[\tau_i(\hat{\pi}) = f(\pi)^+]$. Moreover, if $f(\pi) \succ'_i f(\pi')$ then $f(\pi) \hat{\succ}_i f(\pi)'$, otherwise, if $f(\pi') \succ'_i f(\pi)$ then $f(\pi') \hat{\succ}_i f(\pi)$.

(ii) Moreover, for all $x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$ then $x \hat{\succ}_i y$.

The changes above ensure that $\hat{\pi}$ satisfies the conditions in Equation $(*)$ with respect to both pairs of alternatives and is single-peaked.

Step 2: In Step 1, we constructed another profile $\hat{\pi} \in \mathcal{S}(T)^n$ from $\pi$ and $\pi'$ which satisfies Equation $(*)$. Due to the above arguments, $f(\hat{\pi}) \in \mathit{Range}(\hat{\pi})$, which implies that $f(\hat{\pi}) \in \{f(\pi), f(\pi)^+\}$. If $f(\hat{\pi}) = f(\pi)^+$, then by *constitutional consistency*, we have $f(\hat{\pi}) = f(\hat{\pi}_{\{f(\pi),f(\pi)^+\}}) = f(\pi_{\{f(\pi),f(\pi)^+\}}) = f(\pi)$, where the second inequality is due to Equation $(*)$ in our construction and the last inequality is an implication of *constitutional consistency*. But this is a contradiction since $f(\pi) \neq f(\pi)^+$. Therefore, $f(\hat{\pi}) = f(\pi)$.

By *constitutional consistency* and the above observation,

$$f(\hat{\pi}) = f(\hat{\pi}_{\{f(\pi),f(\pi')\}}) = f(\pi_{\{f(\pi),f(\pi')\}}) = f(\pi').$$

The above two equations imply that $f(\pi) = f(\pi')$. This is a contradiction. Therefore, $f(\pi) = f(\pi')$. ■

**Proof of Theorem 1** We prove necessity of the axioms first. It is easy to check that $q$-threshold rules on $T$ are anonymous and unanimous. By applying the results of Bhattacharya (2019), we know that $q$-threshold rules are *constitutionally consistent* on any extremal path $[r, y]$. We prove this property for any profile $\pi \in \mathcal{S}(T)^n$. By property (iii) of $q$-threshold rule, there exists an $x^* \in X$ for every given profile such that $f_r^q|_{[a,b]}(\pi_{[a,b]}) = x^*$. If $x^* \in Range(\pi)$ then $f_r^q(\pi) = x^*$.

We prove sufficiency first. Suppose $f$ is an s.c.f. on $T$ which is *constitutionally consistent*, *unanimous* and *anonymity*. We show that $f$ is a $q$-threshold rule on $T$. We fix an alternative on a terminal node, say $r$. By the results in Bhattacharya (2019), on any path $[r, y] \in \mathbf{E}_r$ for some $y \in X \setminus \{r\}$ the restriction of the s.c.f. $f$ on path $[r, y]$ (denoted $f|_{[r,y]}$) is *constitutionally consistent*, *unanimous* and *anonymous* if and only if it is a *$q$-threshold rule* on $[r, y]$. Let us denote this s.c.f. as $f^q_{[r,y]}$. By the above arguments, $f|_{[r,y]} = f^q_{[r,y]}$ for all $y \in X \setminus \{r\}$ such that $[r, y] \in \mathbf{E}_r$.

An important property needs to be satisfied to ensure that the rules are consistent across different paths. We will call this property *intersectionality*, which is defined below.

**Definition 2** *Suppose $f_{[a,b]}$ and $f_{[a,d]}$ are two $q$-threshold rules on two extremal paths $[a, b]$ and $[a, d]$ respectively. They are said to be* ***intersectional*** *if for any $\pi \in \mathcal{S}(T)^n$,*

$$\Big[f_{[a,b]}(\pi_{[a,b]}) = x' \notin \big([a, b] \cap [a, d]\big)\Big] \Rightarrow \Big[f_{[a,d]}(\pi_{[a,d]}) \in \big([a, b] \cap [a, d]\big)\Big].$$

*Intersectionality* states that at least one of the outcomes of the two s.c.f.s defined on their respective extremal paths must lie in the intersection of the two paths when the profile $\pi$ is restricted to the relevant path.

Take any $\pi \in \mathcal{S}(T)^n$. We show that the every pair of restrictions of the rule $f$ to extremal paths in $\mathbf{E}_r$ are *intersectional*. Suppose $[a,b]$ and $[a,d]$ are two distinct extremal paths. If $\tau(\pi_{[a,b]}) = \tau(\pi_{[a,d]})$, then by definition of restriction of a rule and by *tops-only* property, $f|_{[a,b]}(\pi_{[a,b]}) = f|_{[a,d]}(\pi_{[a,d]})$. In this case, our claim follows directly.

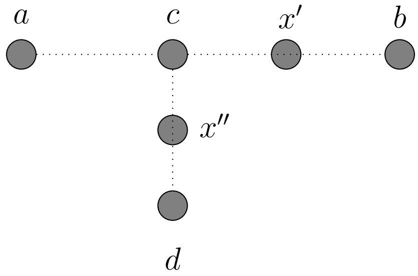


Figure 3: Illustration for Proof of Theorem 1 part (ii)

Suppose $\tau(\pi_{[a,b]}) \neq \tau(\pi_{[a,d]})$ and $f|_{[a,b]}(\pi_{[a,b]}) = x' \notin [a,b] \cap [a,d]$, and assume for contradiction that $f|_{[a,d]}(\pi_{[a,d]}) = x'' \notin [a,b] \cap [a,d]$. By the *tops-only* property, we know that $f|_{[a,d]}(\pi_{[a,d]}) \in \mathit{Range}(\pi_{[a,d]})$. Therefore, it must be that $x' \in [c,b]$ (as illustrated in Figure 3) where $c$ is the last alternative away from $a$ which is both in $[a,b]$ and $[a,d]$.

Now consider $\pi_{\{x',x''\}}$. Note that $\mathit{Range}(\pi_{\{x',x''\}}) = [x',x'']$ and $f|_{[d,b]}(\pi_{\{x',x''\}}) \in \mathit{Range}(\pi_{\{x',x''\}}) = [x',x'']$. Suppose $f|_{[d,b]}(\pi_{\{x',x''\}}) = \tilde{x}$. Then either $\tilde{x} \in [a,b]$ or $\tilde{x} \in [a,d]$. We show that $\tilde{x} \in \{x',x''\}$. Suppose for contradiction that $\tilde{x} \in [a,d] \setminus \{x',x''\}$. By *constitutional consistency*, $f|_{[a,d]}(\pi_{[a,d]}) = x''$ implies that $f|_{[a,d]}(\pi_{\{\tilde{x},x''\}}) = x''$. This is a contradiction since $\tilde{x} \neq x''$. Similar contradiction is obtained if $\tilde{x} \in [a,b]$ Therefore, $f|_{[d,b]}(\pi_{\{x',x''\}}) \in \{x',x''\}$. Suppose $f|_{[d,b]}(\pi_{\{x',x''\}}) = x'$ without loss of generality.

We construct the following profile $\pi' = (\succ'_i)_{i\in N} \in \mathcal{S}(T)^n$ to obtain a contradiction to our initial assumption. Let $\tau_i(\pi') = \tau_i(\pi_{[a,b]}))$ for all $i \in N$. Also,

$$x' \succ_i x'' \implies x' \succ'_i x'' \text{ and } x'' \succ_i x' \implies x'' \succ'_i x'.$$

We argue that the above can be done without affecting the earlier step where we ensured that $\tau_i(\pi') = \tau_i(\pi_{[a,b]}))$ due to the following observations:

(a) Any voter who preferred $x'$ over $x''$ in $\pi$ must be such that either $x' \in$

$[x'', \tau_i(\pi)]$, in which case $\tau_i(\pi_{[a,b]}) \in [x', b]$, or has a peak $\tau_i(\pi)$ such that $x' \notin [\tau_i(\pi), x'']$ in which case $\tau_i(\pi_{[a,b]}) \in [a, x')$. In either case, for these voters their tops will be unaffected in the previous step.

(b) Any voter who preferred $x''$ over $x'$ in $\pi$ must be such that either $x'' \in [\tau_i(\pi), x']$ in which case $\tau_i(\pi_{[a,b]}) = c$ or $x'' \notin [\tau_i(\pi), x']$. In either case, for these voters $\tau_i(\pi_{[a,b]}) \in [a, b]$. Therefore, for these voters too their tops will be unaffected in the previous step.

In all the cases, there is no constraint while constructing $\pi'$ when ensuring that preferences of each voter over $x'$ and $x''$ are the same as they were in $\pi$. Therefore, by our construction, $\pi_{\{x',x''\}} = \pi'_{\{x',x''\}}$.

Similarly, we construct another profile $\pi''$ such that $\tau_i(\pi'') = \tau_i(\pi_{[a,d]})$ for all $i \in N$. We also ensure that,

$$x' \succ_i x'' \implies x' \succ''_i x'' \text{ and } x'' \succ_i x' \implies x'' \succ''_i x'.$$

By *tops-onlyness*, we have $f|_{[a,b]}(\pi_{[a,b]}) = f(\pi') = x'$ and $f|_{[a,d]}(\pi_{[a,d]}) = f(\pi'') = x''$. By *constitutional consistency*, we have $f(\pi) = f|_{[b,d]}(\pi'_{\{x',x''\}}) = x'$ and $f(\pi'') = f|_{[b,d]}(\pi'_{\{x',x''\}}) = x''$. This is a contradiction.

(iii) We show that *intersectionality* implies the following: for all $x \in T$ such that $deg(x) \geq 3$, $q_x^{[r,y]} + q_x^{[r,y']} < n + 2$ for all extremal paths $[r, y], [r, y'] \in \mathbf{E}_r$.

We first prove for even number of voters. Suppose for contradiction that the above condition is violated. Then, there exists a node $x$ which has degree greater than or equal to 3 which belongs to two distinct extremal paths $[r, y]$ and $[r, y']$ in $\mathbf{E}_r$, and $q_x^{[r,y]} + q_x^{[r,y']} \geq n + 2$ as shown in Figure 6 below. This implies that there exists an integer $k \in \{1, 2, ..., n\}$ such that $q_x^{[r,y]} + q_x^{[r,y']} - (k + 1) = n$.

Consider the following profile with two types of preferences with the top three alternatives as follows:

$$\pi = \left[ \begin{bmatrix} x(y') \\ x \\ x(y) \\ \vdots \end{bmatrix}^{q_x^{[r,y]}-1} \quad \begin{bmatrix} x(y) \\ x \\ x(y') \\ \vdots \end{bmatrix}^{q_x^{[r,y']}-k} \right]$$

where $x(y)$ and $x(y')$ are the alternatives adjacent to $x$ away from $r$ in the

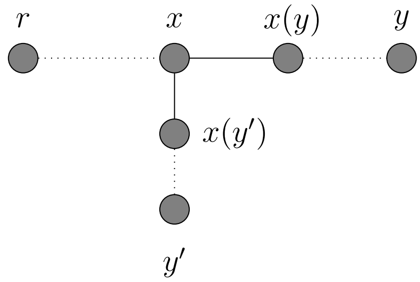


Figure 4

path $[r, y]$ and $[r, y']$ respectively and the preferences over other alternatives can be defined in any way consistent with single-peakedness. By definition of $q$-threshold rules on the path $[r, y]$ and $[r, y']$ we have,

$$f^q_{[r,y]}(\pi_{\{x,x(y)\}}) = f^q_{[r,y]}(x^{q_x^{[r,y]-1}}, x(y)^{q_x^{[r,y']}-k}) = x(y)$$

$$f^q_{[r,y]}(\pi_{\{x,x(y')\}}) = f^q_{[r,y']}(x^{q_x^{[r,y']-k}}, x(y')^{q_x^{[r,y]-1}}) = x(y')$$

The two equations are due to the fact that $x$ does not have enough votes at the top to beat the other alternative since its threshold is strictly greater than its top votes. This is a contradiction to *intersectionality*. Therefore, $q_x^{[r,y]} + q_x^{[r,y']} < n + 2$ for any two paths $[r, y], [r, y'] \in \mathbf{E}_r$ such that $x \in [r, y] \cap [r, y']$. ■

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