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Abstract

In this paper, we consider auctions with resale when bidders are symmetric and draw correlated signals. We show that the all-pay second-price auction with resale generates the highest expected revenue for the seller among the family of first-price auction, second-price auction, all-pay first-price auction, all-pay second-price auction, first-price auction with resale, second-price auction with resale, first-price all-pay auction with resale, and second-price all-pay auction with resale.

JEL classification: D44, D82

Keywords: all-pay auction, war-of-attrition, resale, time delay, affiliation, correlated signals, revenue

1 Introduction

In this paper, we consider resale possibilities in auctions with symmetric bidders and correlated signals. The symmetric assumption ensures that the allocation of the object is efficient under standard sealed-bid auctions, i.e., the object is allocated to the highest value bidder. Despite efficiency, the bidders benefit from existence of a resale market because of a time delay between auction and resale. The time delay disproportionately reduces bidders' values. The winner's value declines because he consumes the object before resale occurs while the loser's value declines because the object is depleted. To the best of my knowledge, this is the first paper to consider correlated values in auctions with resale, where the set of bidders is same during the auction and resale.

Consider two risk neutral bidders who draw their valuations from correlated probability distributions for one indivisible object. The game is played for two dates. At Date 0, a sealed bid auction occurs. The winner of the auction utilizes the object for a fixed amount of time and induces utility from it. However, in the process, he depletes the object

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which reduces the loser's valuation. At Date 1, a resale trade occurs which succeeds with a positive probability. The valuations of the bidders become common knowledge before date 1. Date 0 is called bid date, Date 1 is called resale date and the time between the two dates is called interim date.

By considering a family of trade rules during the resale date where the market power is divided between the two bidders, the equilibrium has been characterized with analytical solutions for first-price auction with resale, second-price auction with resale, all-pay first-price auction with resale and all-pay second-price auction with resale.

Next, we pairwise compare the seller's expected revenue between first-price auction with resale, second-price auction with resale, all-pay first-price auction with resale and all-pay second-price auction with resale. We show that (a) the seller generates more expected revenue from the second-price auction with resale than the first-price auction with resale, (b) the seller generates more expected revenue from the all-pay second-price auction with resale than the all-pay first-price auction with resale, (c) the seller generates more expected revenue from the all-pay first-price auction with resale than the first-price auction with resale, and (d) the seller generates more expected revenue from the all-pay second-price auction with resale than the second-price auction with resale.

Consequently, the all-pay second-price auction with resale generates the highest revenue, the second-price auction with resale generates the second highest revenue, the all-pay first-price auction with resale generates the third highest revenue and the all-pay first-price auction with resale generates the fourth highest revenue.

We have also compared the seller's expected revenue between our model and other standard models in the literature. In particular, we consider two standard models: Milgrom and Weber [14] and Krishna and Morgan [13]. Milgrom and Weber [14] study first- and second-price auction without resale while Krishna and Morgan [13] study all-pay first- and all-pay second-price auction without resale. We show that, if the market power with the winner is sufficiently high, then (a) the seller generates more expected revenue from the first-price auction with resale than the first-price auction without resale, (b) the seller generates more expected revenue from the second-price auction with resale than the second-price auction without resale, (c) the seller generates more expected revenue from the all-pay first-price auction with resale than the all-pay first-price auction without resale, and (d) the seller generates more expected revenue from the all-pay second-price auction with resale than the all-pay second-price auction without resale.

Consequently, the all-pay second-price auction with resale generates the highest expected revenue for the seller among the family of first-price auction without resale, second-price auction without resale, all-pay first-price auction without resale, all-pay second-price auction without resale, first-price auction with resale, second-price auction with resale, all-pay first-price auction with resale and all-pay second-price auction with resale.

1.1 The literature

Auctions with resale and independent private valuations have been studied in Gupta and Lebrun [9], Hafalir and Krishna [10], Virág [16, 17], and Khurana [11, 12] among others. Hafalir and Krishna [10] show that the first-price auction is revenue superior to the second-price auction. Khurana [12] is the first paper to consider time delays in asymmetric auctions with resale. The paper shows that the first-price auction is revenue superior to the second-price auction for the special case of uniform distributions. Khurana [11] considers time delays in symmetric auctions with resale and studies the impact of information, that concerns about revealing values and bids after the auction, on the bid behavior and the seller's expected revenue. The paper also establishes revenue equivalence principle under complete information where all the bids and values are revealed after the auction.

All-pay auctions under complete information have been examined in Baye et al. [2, 3], Che and Gale [5], Gelder et al. [7], and Georgiadis et al. [8] among others. All-pay auctions under incomplete information have been examined in Amann and Leininger [1], Krishna and Morgan [13], Fibich et al. [6], Seel [15], and Betto and Thomas [4] among others.

Amann and Leininger [1] characterize and prove existence of an equilibrium by considering two asymmetric bidders. Betto and Thomas [4] consider that the actions of bidders impact not only the winning probability of other bidders but also the value of the prize for other bidders. They show existence of a unique equilibrium in mixed strategies. Fibich et al. [6] consider risk averse bidders and study numerous comparative results. Krishna and Morgan [13] show that the second-price all-pay auction is revenue superior to all other standard sealed-bid auctions by considering correlated valuations among the bidders.

The structure of the paper is as follows. In Section 2, we setup the model. In Section 3, we characterize an equilibrium for different auction formats. In Section 4, we provide revenue ranking results. Section 5 concludes. The proofs are relegated to the appendix.

2 Economic environment

Consider a seller wishes to allocate an indivisible object via a sealed bid auction. Two bidders with risk neutral preferences and private values participate in the auction. The valuation space for both the bidders is $T = [0,1]$. Let the set of bidders be $N = \{1,2\}$. Nature draws a valuation profile $\mathbf{t} = (t_1, t_2) \in T^2$ from a joint probability distribution and informs them privately. Let random variables be denoted by $\mathcal{T}_1$ and $\mathcal{T}_2$. The joint probability distribution is $F : T^2 \to \Re_+$ and the associated density function is $f : T^2 \to \Re_+$.

The probability distribution is symmetric, i.e., $F(t_1, t_2) = F(t_2, t_1)$ for every $(t_1, t_2), (t_2, t_1) \in T^2$. We assume that F is twice continuously differentiable and f is bounded away from zero. Since a bidder chooses an action after observing his own value, he operates on the conditional probability distribution which is denoted by $G : T^2 \to \Re_+$. Given a bidder with value t, $G(., t)$ gives the probability distribution about his opponent's value. Let the associated conditional density function be $g : T^2 \to \Re_+$.

Given $\mathbf{t}, \mathbf{v} \in T^2$, let $\mathbf{t} \vee \mathbf{v} := (\sup\{t_1, v_1\}, \sup\{t_2, v_2\})$ and $\mathbf{t} \wedge \mathbf{v} := (\inf\{t_1, v_1\}, \inf\{t_2, v_2\})$. The expression $\mathbf{t} \vee \mathbf{v}$ is called the join of $\mathbf{t}$ and $\mathbf{v}$ while $\mathbf{t} \wedge \mathbf{v}$ is called the meet of $\mathbf{t}$ and $\mathbf{v}$.

Definition 1. *We say that the random variables $\mathcal{T}_1$ and $\mathcal{T}_2$ are* affiliated *if for every $\mathbf{t}, \mathbf{v} \in T^2$, we have*[1]

$$f(\mathbf{t} \vee \mathbf{v}) f(\mathbf{t} \wedge \mathbf{v}) > f(\mathbf{t}) f(\mathbf{v})$$

In words, if bidder 1 (resp., bidder 2) draws a higher value, then his belief about bidder 2 (resp., bidder 1) drawing a higher value rises.

The following is a standard result in the literature.

Lemma 1. *If the random variables $\mathcal{T}_1$ and $\mathcal{T}_2$ are affiliated, then the following holds:*

1. *For every $\mathbf{t}, \mathbf{v} \in T^2$ such that $\mathbf{t} \gg \mathbf{v}$, we have*

$$\frac{g(v_2, t_1)}{g(v_2, v_1)} < \frac{g(t_2, t_1)}{g(t_2, v_1)}$$

2. *For every $t_1, v_1 \in T$ such that $t_1 > v_1$, we have $g(., t_1)/g(., v_1)$ is strictly increasing.*
3. *For every $t_1, v_1 \in T$ such that $t_1 > v_1$, we have*

$$\frac{g(t_2, t_1)}{1 - G(t_2, t_1)} < \frac{g(t_2, v_1)}{1 - G(t_2, v_1)}$$

[1] We say f is *supermodular* if for every $\mathbf{t}, \mathbf{v} \in T^2$, we have $f(\mathbf{t} \vee \mathbf{v}) + f(\mathbf{t} \wedge \mathbf{v}) > f(\mathbf{t}) + f(\mathbf{v})$. Therefore, $\mathcal{T}_1$ and $\mathcal{T}_2$ are affiliated if and only if $\ln f$ is supermodular.

4. For every $t_1, v_1 \in T$ such that $t_1 > v_1$, we have

$$\frac{g(t_2, t_1)}{G(t_2, t_1)} > \frac{g(t_2, v_1)}{G(t_2, v_1)}$$

5. For every $t_1, v_1 \in T$ such that $t_1 > v_1$, we have $G(t_2, t_1) < G(t_2, v_1)$.

Property 2 is called the monotone likelihood ratio property, Property 3 is called the hazard rate dominance, Property 4 is called the reverse hazard rate, and Property 5 is called the first-order stochastic dominance.

Assumption 1. *The random variables $\mathcal{T}_1$ and $\mathcal{T}_2$ are affiliated.*

The game is designed as follows. Consider a two-date game whereat Date 0, a sealed-bid auction is being held by the owner of the object. After Date 0 and before Date 1, (a) the winner of Date 0 consumes the object for a fixed period of time and derives value from it, while the loser loses some value as the winner depletes the object in the process of consumption, and (b) the owner of the object reveals all the values. The game proceeds to Date 1 where both the bidders may engage in a resale via a "trade rule".

Date 0 is called the *bid date*, Date 1 is called the *resale date*, and the period between the two dates is called the *interim date*. The interim date and the trade rule are exogenous.

During the interim date, the winner incurs value according to a fraction α_R while the loser loses according to a fraction α_B. To elaborate, if the winner draws a value w, then he derives a value of $\alpha_R w$ during the interim date and if the loser draws a value of l, then he loses $\alpha_B l$ during the interim date. To ease up the exposition, α_R will be called the *consumption rate* and α_B will be called the *depletion rate*.

At Date 1, bidders act according to their *ex-post values* – the values at the time of resale. Given a value w of the winner, his ex-post value is $(1 - \alpha_R)w$, as he has already exhausted $\alpha_R w$ worth of object. Given a value l of the loser, his ex-post value is $(1 - \alpha_B)l$, as he has already lost $\alpha_B l$ worth of object due to its depletion by the winner. Consequently, the winner accepts a resale offer as long as the resale price exceeds his ex-post value while the loser accepts a resale offer as long as the resale price is lower than his ex-post value. Therefore, a trade is successful if and only if the ex-post value of the winner is lower that the ex-post value of the loser.

In this paper, we consider four different sealed bid auctions: (a) first-price auction, (b) second-price auction, (c) all-pay first-price auction, and (d) all-pay second-price auction (war-of-attrition). The first-price auction has been characterized in Subsection 3.1, second-price auction has been characterized in Subsection 3.2, all-pay first-price auction has been characterized in Subsection 3.3, and all-pay second-price auction has been characterized in Subsection 3.4.

Denote the bid symmetric function in the first-price auction by σ^1, second-price auction by σ^2, first-price all-pay auction by σ^1_*, and second-price all-pay auction by σ^2_*. Let's restrict to the family of bid functions that are strictly monotone, continuous, and onto. Therefore, the inverse bid functions exist which are denoted by π^1, π^2, π^1_* and π^2_* for their counterparts.

The family of trade rule is defined as

$$p(w,l) = \lambda_w w + \lambda_l l \tag{1}$$

where w is the winner's value, l is the loser's value, and λ_w and λ_l are exogenous parameters specific to the winner and loser respectively.

Assumption 2. *The following is true:*

1. $\alpha_R > \alpha_B$*,*
2. $\max\{\alpha_R + \lambda_w, \alpha_B + \lambda_l\} < 1$*,*
3. $k < 1$*, where*

$$k := \max\left\{\frac{\lambda_w}{1-\alpha_B-\lambda_l}, \frac{1-\alpha_R-\lambda_w}{\lambda_l}\right\}$$

4. $\lambda_w + \lambda_l + \alpha_R + \alpha_B > 1$*,*
5. $\lambda_w, \lambda_l > 0$*.*

The first assumption ensures that there are always expected potential profits from resale, i.e., the winner's value declines faster than the loser's value. The rest of the assumptions ensure tractability of the model.

3 Characterization results

In this section, we characterize an equilibrium of the first-price auction with resale, second-price auction with resale, all-pay first-price auction with resale and all-pay second-price auction with resale.

3.1 First-price auction with resale

Consider the first-price auction. Let the expected utility function be $U^1 : T \times \Re_+ \to \Re$. The underneath proposition records the expected utility function.

Proposition 1. *Let Assumptions 1 and 2 be satisfied. Given a value* $t \in T$ *and a bid* $b \in \Re_+$*, the expected utility function of a bidder is*

$$\begin{aligned} U^1(t,b) = G(kt,t)(t-b) &+ \int_{kt}^{\pi^1(b)} (\alpha_R t + \lambda_w t + \lambda_l x - b) g(x,t) \mathrm{d}x \\ &+ \int_{\pi^1(b)}^{t/k} [(1-\alpha_B-\lambda_l) - \lambda_w x] g(x,t) \mathrm{d}x \end{aligned} \tag{2}$$

To ease up the notation, denote $\alpha = (\alpha_R, \alpha_B)$, $\lambda = (\lambda_w, \lambda_l)$, and

$$\Gamma(\alpha, \lambda) = 2\lambda_w + 2\lambda_l + \alpha_R + \alpha_B - 1$$

The parameter $\Gamma(\alpha, \lambda)$ is interpreted as an indicator of reseller's market power, i.e., a higher value of Γ indicates that a higher market power lies with the reseller.

In the following result, we characterize an equilibrium of the first-price auction with resale.

Proposition 2. *Let Assumptions 1 and 2 be satisfied. The function σ^1 is a symmetric perfect Bayesian equilibrium of the first-price auction if and only if*

$$\sigma^1(t) = \Gamma(\alpha, \lambda) \int_0^t \exp\left\{ - \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x \right\} y \frac{g(y,y)}{G(y,y)} \mathrm{d}y \tag{3}$$

for every $t \in T$.

The above proposition provides a formula for bid function in a first-price auction for a generalized family of probability distributions. If the market power with the winner (resp., loser) is sufficiently high, bidders raise (resp., lower) their bids.

If $\Gamma(\alpha, \lambda)$ approaches 1, the gains from trade disappear and the bid function approaches first-price auction without resale that is given in Milgrom and Weber [14].

3.2 Second-price auction with resale

Consider the second-price auction. Let the expected utility function be $U^2 : T \times \Re_+ \to \Re$. The underneath proposition records the expected utility function.

Proposition 3. *Let Assumptions 1 and 2 be satisfied. Given a value t and a bid b of a bidder, the expected utility function is*

$$\begin{aligned} U^2(t,b) = &\int_0^{kt} [t - \sigma^2(x)] g(x,t) \mathrm{d}x + \int_{kt}^{\pi^2(b)} [\alpha_R t + \lambda_w t + \lambda_l x \\ &- \sigma^2(x)] g(x,t) \mathrm{d}x + \int_{\pi^2(b)}^{t/k} [(1 - \alpha_B - \lambda_l)t - \lambda_w x] g(x,t) \mathrm{d}x \end{aligned} \tag{4}$$

In the following result, we characterize an equilibrium of the second-price auction with resale.

Proposition 4. *Let Assumptions 1 and 2 be satisfied. The function σ^2 is a perfect Bayesian equilibrium of the second-price auction if and only if*

$$\sigma^2(t) = \Gamma(\alpha, \lambda)t \tag{5}$$

for every $t \in T$.

It turns out that if $\Gamma(\alpha, \lambda) > 1$, bidders outbid their values; if $\Gamma(\alpha, \lambda) < 1$, bidders shade their values; and if $\Gamma(\alpha, \lambda)$ approaches 1, all the gains from trade disappear and bidders bid their values. Importantly, the bid function does not rely on the choice of probability distributions, i.e., it is robust to a bidder's belief about his opponent.

Intuitively, if market power with the winner is sufficiently high, bidders have an incentive to raise the probability of winning and thus they bid higher.

In the next result, we compare bid functions between the first- and second-price auction with resale.

Proposition 5. *Let Assumptions 1 and 2 be satisfied. Let $g(t,t) > G(t,t)$ for every $t \in (0,1)$. Then, for every $t \in (0,1)$, we have*

$$\sigma^1(t) > \sigma^2(t)$$

The above result says that bidders bid higher under the first-price auction than under the second-price auction.

3.3 All-pay first-price auction with resale

Consider the first-price all-pay auction. Let the expected utility function be $U^1_* : T \times \Re_+ \to \Re$. In the underneath proposition, we record the expected utility function.

Proposition 6. *Let Assumptions 1 and 2 be satisfied. Given a value t and a bid b of a bidder, the expected utility function is*

$$\begin{aligned} U^1_*(t,b) = G(kt,t)t &+ \int_{kt}^{\pi^1_*(b)} (\alpha_R t + \lambda_w t + \lambda_l x) g(x,t) \mathrm{d}x \\ &+ \int_{\pi^1_*(b)}^{t/k} [(1 - \alpha_B - \lambda_l)t - \lambda_w x] g(x,t) \mathrm{d}x - b \end{aligned} \tag{6}$$

In the following result, we characterize an equilibrium of the first-price all-pay auction with resale.

Proposition 7. *Let Assumptions 1 and 2 be satisfied. Let $g(y,.)$ be strictly increasing for every $y \in \Re_+$. Then, σ^1_* is a perfect Bayesian equilibrium of the first-price all-pay auction if and only if*

$$\sigma^1_*(t) = \Gamma(\alpha, \lambda) \int_0^t x g(x,x) \mathrm{d}x \tag{7}$$

for every $t \in T$.

If $\Gamma(\alpha,\lambda)$ approaches 1, the gains from trade disappear and the bid function approaches all-pay first-price auction without resale that is given in Krishna and Morgan [13].

3.4 All-pay second-price auction with resale

Consider the all-pay auction second-price which is also called a war-of-attrition. Let the expected utility function be $U_*^2 : T \times \Re_+ \to \Re$. In the underneath proposition, we record the expected utility function.

Proposition 8. *Let Assumptions 1 and 2 be satisfied. Given a value t and a bid b of a bidder, the expected utility function is*

$$\begin{aligned} U_*^2(t,b) = \int_0^{kt} [t - \sigma_*^2(x)] g(x,t)\mathrm{d}x + \int_{kt}^{\pi_*^2(t)} [\alpha_R t + \lambda_w t + \lambda_l x \\ - \sigma_*^2(x)] g(x,t)\mathrm{d}x + \int_{\pi_*^2(b)}^{t/k} [(1-\alpha_B-\lambda_l)t - \lambda_w x - b] g(x,t)\mathrm{d}x \\ - b[1 - G(t/k,t)] \end{aligned} \tag{8}$$

Proposition 9. *Let Assumptions 1 and 2 be satisfied. Let σ_*^2 be a perfect Bayesian equilibrium of the second-price all-pay auction. Then,*

$$\sigma_*^2(t) = \Gamma(\alpha,\lambda)\int_0^t x \frac{g(x,x)}{1-G(x,x)}\mathrm{d}x \tag{9}$$

for every $t \in T$.

If $\Gamma(\alpha,\lambda)$ approaches 1, the gains from trade disappear and the bid function approaches all-pay second-price auction without resale that is given in Krishna and Morgan [13].

In the next result, we compare bid functions between the all-pay first- and all-pay second-price auction with resale.

Proposition 10. *Let Assumptions 1 and 2 be satisfied. For every $t \in (0,1)$, we have*

$$\sigma_*^2(t) > \sigma_*^1(t)$$

The above result says that bidders bid higher under all-pay second-price auction than under all-pay first-price auction.

4 Comparative results

In this section, we present two main results of the paper.

In the following result, we pairwise compare the seller's expected revenue between the first-price auction with resale, second-price auction with resale, all-pay first-price auction with resale, and all-pay second-price auction with resale.

Theorem 1. *Let Assumptions 1 and 2 be satisfied. Let $R^1(\sigma^1)$ be the seller's* ex-ante *expected revenue in the first-price auction with resale. Let $R^2(\sigma^2)$ be the seller's* ex-ante *expected revenue in the second-price auction with resale. Let $R^1(\sigma^1_*)$ be the seller's* ex-ante *expected revenue in the all-pay first-price auction with resale. Let $R^2(\sigma^2_*)$ be the seller's* ex-ante *expected revenue in the all-pay second-price auction with resale. Then,*

1. $R^2(\sigma^2_*) > R^1(\sigma^1_*)$
2. $R^1(\sigma^1_*) > R^1(\sigma^1)$
3. $R^2(\sigma^2_*) > R^2(\sigma^2)$

If $g(y,.)$ be strictly increasing for every $y \in \Re_+$,

4. $R^2(\sigma^2) > R^1(\sigma^1)$
5. $R^2(\sigma^2) > R^1(\sigma^1_*)$

Part 1 shows that the all-pay second-price auction with resale revenue dominates the all-pay first-price auction with resale. Part 2 shows that the all-pay first-price auction with resale revenue dominates the first-price auction with resale. Part 3 shows that the all-pay second-price auction with resale revenue dominates the second-price auction with resale. Part 4 shows that the second-price auction with resale revenue dominates the first-price auction with resale. Part 5 shows that the second-price auction with resale revenue dominates the all-pay first-price auction with resale.

A revenue ranking principle among the first-price auction with resale, second-price auction with resale, first-price all-pay auction with resale and second-price all-pay auction with resale is recorded in the following remark.

Remark 1. *Let Assumptions 1 and 2 be satisfied. Let $g(y,.)$ be strictly increasing for every $y \in \Re_+$. Then,*

$$R^2(\sigma^2_*) > R^2(\sigma^2) > R^1(\sigma^1_*) > R^1(\sigma^1)$$

It says that all-pay second-price auction with resale revenue dominates all-pay first-price auction with resale, first-price auction with resale and second-price auction with resale.

We now compare the seller's expected revenue of the present paper with two standard models in the literature: Milgrom and Weber [14] (henceforth, M-W) and Krishna and Morgan [13] (henceforth, K-M). These two papers consider correlated signals without the possibility of resale. The M-W paper considers the first- and second-price auction and the K-P paper considers the all-pay first- and all-pay second-price auction.

M-W show that the second-price auction without resale revenue dominates the first-price auction without resale. K-P show that the ranking provided in Remark 1 holds for auctions without resale. They also show that the all-pay second-price auction without resale performs better than all-pay first-price auction without resale, first-price auction without resale and second-price auction without resale in terms of expected revenue.

The next result captures the impact of resale on the M-W and K-P models from the seller's point of view. Denote the bid function in the first-price auction without resale by β^1, the second-price auction without resale by β^2, the all-pay first-price auction without resale by β^1_* and the all-pay second-price auction without resale by β^2_*.

Theorem 2. *Let Assumptions 1 and 2 be satisfied and let* $\Gamma(\alpha, \lambda) > 1$. *Let* $R^1(\beta^1)$ *be the seller's* ex-ante *expected revenue in the first-price auction without resale. Let* $R^2(\beta^2)$ *be the seller's* ex-ante *expected revenue in the second-price auction without resale. Let* $R^1(\beta^1_*)$ *be the seller's* ex-ante *expected revenue in the all-pay first-price auction without resale. Let* $R^2(\beta^2_*)$ *be the seller's* ex-ante *expected revenue in the all-pay second-price auction without resale. Then,*

1. $R^1(\sigma^1) > R^1(\beta^1)$
2. $R^2(\sigma^2) > R^2(\beta^2)$
3. $R^1(\sigma^1_*) > R^1(\beta^1_*)$
4. $R^2(\sigma^2_*) > R^2(\beta^2_*)$

Part 1 shows that the seller induces higher expected revenue under the first-price auction with resale than the first-price auction without resale. Part 2 shows that the seller induces higher expected revenue under the second-price auction with resale than the second-price auction without resale. Part 3 shows that the seller induces higher expected revenue under the all-pay first-price auction with resale than the all-pay first-price auction without resale. Part 4 shows that the seller induces higher expected revenue under the all-pay second-price auction with resale than the all-pay second-price auction without resale.

From K-M, we know that the all-pay second-price auction revenue dominates the all-pay first-price auction, first-price auction and second-price auction. From Theorem 2 and Remark 1, we infer that:

Remark 2. *The all-pay second-price auction with resale revenue dominates the first-price auction without resale, second-price auction without resale, all-pay first-price auction without resale, all-pay second-price auction without resale, first-price auction with resale, second-price auction with resale and all-pay first-price auction with resale.*

5 Conclusion

In this paper, we have considered correlated signals in auctions with resale. Our two main results are: (a) the seller's expected revenues are highest in the all-pay second-price auction with resale followed by the second-price auction with resale followed by the all-pay first-price auction with resale followed by the first-price auction with resale and (b) the all-pay second-price auction with resale outperforms all first-price auction with and without resale, second-price auction with and without resale, all-pay first-price auction with and without resale and all-pay second-price auction without resale.

A Appendix: Proofs

Proof of Proposition 1. Consider a bidder who draws a value t and bids b while the other bidder bids $\sigma^1(\mathcal{T})$. He wins if and only if $b > \sigma^1(\mathcal{T})$ which is equivalent to $\mathcal{T} < \pi^1(t)$. Upon win, a trade ensues if and only if the ex-post value of the winner does not exceed the resale price and the ex-post value of the loser exceeds the resale price, i.e., $(1-\alpha_R)t < \lambda_w t + \lambda_l \mathcal{T} < (1-\alpha_B)\mathcal{T}$. This implies

$$\mathcal{T} > \frac{1-\alpha_R-\lambda_w}{\lambda_l}t \quad \text{and} \quad \mathcal{T} > \frac{\lambda_w}{1-\alpha_B-\lambda_l}t$$

which is equivalent to

$$\mathcal{T} > \max\left\{\frac{1-\alpha_R-\lambda_w}{\lambda_l}, \frac{\lambda_w}{1-\alpha_B-\lambda_l}\right\}t = kt$$

Therefore, with probability $\mathcal{T} < kt$, trade does not ensue and the bidder incurs a utility of $t-b$, as he retains the object. On the other hand, with probability $\mathcal{T} > kt$, bidder realizes a utility of $\alpha_R t + \lambda_w t + \lambda_l \mathcal{T} - b$.

The bidder loses if and only if $\mathcal{T} > \pi^1(b)$. Upon losing, a trade ensues if and only if $(1-\alpha_B)t > \lambda_w \mathcal{T} + \lambda_l t > (1-\alpha_R)\mathcal{T}$. This implies

$$\mathcal{T} < \frac{1-\alpha_B-\lambda_l}{\lambda_w}t \quad \text{and} \quad \mathcal{T} < \frac{\lambda_l}{1-\alpha_R-\lambda_w}t$$

which equals

$$\mathcal{T} < \min\left\{\frac{1-\alpha_B-\lambda_l}{\lambda_w}, \frac{\lambda_l}{1-\alpha_R-\lambda_w}\right\}t = \frac{t}{k}$$

Therefore, with probability $\pi^1(b) < \mathcal{T} < t/k$, the bidder incurs a utility of $(1-\alpha_B)t - \lambda_w \mathcal{T} - \lambda_l t$. On the other hand, with probability $\mathcal{T} > t/k$, trade does not ensue and the bidder realizes a utility of 0.

Thus, the expected utility function is given by (2). ■

Proof of Proposition 2. Let σ^1 be a symmetric perfect Bayesian equilibrium. Differentiating (2) w.r.t. b, we have

$$\begin{aligned} \mathrm{D}_b U^1(t,b) = \mathrm{D}\pi^1(b) g(\pi^1(b),t)[(\alpha_R+\alpha_B+\lambda_w+\lambda_l-1)t \\ + (\lambda_w+\lambda_l)\pi^1(b) - b] - G(\pi^1(b),t) \end{aligned} \tag{10}$$

In equilibrium, $\mathrm{D}_b U^1(\pi^1(b), b) = 0$ which gives

$$\mathrm{D}\pi^1(b) = \frac{G(\pi^1(b), \pi^1(b))}{g(\pi^1(b), \pi^1(b))} \frac{1}{\Gamma(\alpha, \lambda)\pi^1(b) - b}$$

To find an explicit expression, we apply change-of-variables. Since $b = \sigma^1 \circ \pi^1(b)$, differentiating w.r.t. b gives $1 = \mathrm{D}\pi^1(b)\mathrm{D}\sigma^1 \circ \pi^1(b)$. Applying this in the above equation gives

$$\mathrm{D}\sigma^1 \circ \pi^1(b) = \frac{g(\pi^1(b), \pi^1(b))}{G(\pi^1(b), \pi^1(b))}(\Gamma(\alpha, \lambda)\pi^1(b) - b)$$

Using $\pi^1(b) = t$ in the above equation gives

$$\mathrm{D}\sigma^1(t) = \frac{g(t,t)}{G(t,t)}[\Gamma(\alpha, \lambda)t - \sigma^1(t)]$$

which can be rewritten as

$$\mathrm{D}\sigma^1(t) + \frac{g(t,t)}{G(t,t)}\sigma^1(t) = \Gamma(\alpha, \lambda)t\frac{g(t,t)}{G(t,t)}$$

Let the integrating factor be

$$Z(t,t) = \exp\left\{ - \int_t^1 \frac{g(x,x)}{G(x,x)} \mathrm{d}x \right\}$$

Therefore,

$$\mathrm{D}Z(t,t) = \exp\left\{ - \int_t^1 \frac{g(x,x)}{G(x,x)} \mathrm{d}x \right\} \frac{g(t,t)}{G(t,t)} = Z(t,t)\frac{g(t,t)}{G(t,t)}$$

Now,

$$\begin{aligned} \mathrm{D}[Z(t,t)\sigma^1(t)] &= \mathrm{D}Z(t,t)\sigma^1(t) + \mathrm{D}\sigma^1(t)Z(t,t) \\ &= Z(t,t)\left[\mathrm{D}\sigma^1(t) + \sigma^1(t)\frac{g(t,t)}{G(t,t)}\right] \\ &= \Gamma(\alpha, \lambda)t\mathrm{D}Z(t,t) \end{aligned}$$

Applying the fundamental theorem of calculus with the fact that $\sigma^1(0) = 0$, we have

$$Z(t,t)\sigma^1(t) = \Gamma(\alpha, \lambda) \int_0^t y\mathrm{D}Z(y,y)\mathrm{d}y$$

which equals

$$\begin{aligned} \sigma^1(t) &= \Gamma(\alpha, \lambda) \exp\left\{ \int_t^1 \frac{g(x,x)}{G(x,x)} \mathrm{d}x \right\} \\ &\quad \int_0^t y\frac{g(y,y)}{G(y,y)} \exp\left\{ - \int_y^1 \frac{g(x,x)}{G(x,x)} \mathrm{d}x \right\} \mathrm{d}y \\ &= \Gamma(\alpha, \lambda) \int_0^t \exp\left\{ - \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x \right\} y\frac{g(y,y)}{G(y,y)} \mathrm{d}y \end{aligned}$$

To prove the converse, let σ^1 solve (3). We argue that σ^1 is optimal. To contradict, let a bidder with value t overbid to c where $c > \sigma^1(t)$, i.e., $\pi^1(c) > t$. Rewriting (10), we have

$$\begin{aligned} \mathrm{D}_c U^1(t,c) = G(\pi^1(c),t)\Big\{ \mathrm{D}\pi^1(c) \frac{g(\pi^1(c),t)}{G(\pi^1(c),t)} [(\alpha_R + \alpha_B + \lambda_w + \lambda_l - 1)t \\ + (\lambda_w + \lambda_l)\pi^1(c) - c] - 1 \Big\} \end{aligned}$$

From Property 4 of Lemma 1, we have

$$\begin{aligned} \mathrm{D}_c U^1(t,c) &< G(\pi^1(c),t)\Big\{ \mathrm{D}\pi^1(c) \frac{g(\pi^1(c),\pi^1(c))}{G(\pi^1(c),\pi^1(c))} [\Gamma(\alpha,\lambda)\pi^1(c) - c] - 1 \Big\} \\ &= \mathrm{D}_c U^1(\pi^1(c),c) \\ &= 0 \end{aligned}$$

Thus, overbid is not desirable. By reversing the inequalities, we can argue that underbid is also not desirable. So, π^1 is optimal. ■

Proof of Proposition 3. Consider bidder 1 with value t and bid b and consider bidder 2 who bids $\sigma^2(\mathcal{T})$. Bidder 1 wins if and only if $\mathcal{T} < \pi^2(b)$ and trade succeeds if and only if $\mathcal{T} > kt$. Therefore, with probability $\mathcal{T} < kt$, trade fails which gives bidder 1 a utility of $t - \sigma^2(\mathcal{T})$ and with probability $kt < \mathcal{T} < \pi^2(b)$, trade succeeds which gives him a utility of $(\alpha_R + \lambda_w)t + \lambda_l \mathcal{T} - \pi^2(\mathcal{T})$.

Bidder 1 loses if and only if $\mathcal{T} > \pi^2(b)$ and trade succeeds if and only if $\mathcal{T} < t/k$. Therefore, with probability $\pi^2(b) < \mathcal{T} < t/k$, trade succeeds which gives bidder 1 a utility of $(1-\alpha_B-\lambda_l)t - \lambda_w \mathcal{T}$ and with probability $\mathcal{T} > t/k$, trade fails which gives him a utility of 0.

Thus, the expected utility function can be written as (4). ■

Proof of Proposition 4. Let σ^2 be an equilibrium. From (4), we have

$$\begin{aligned} \mathrm{D}_b U^2(t,b) = \mathrm{D}\pi^2(b) g(\pi^2(b),t)[(\lambda_w + \lambda_l + \alpha_R + \alpha_B - 1)t \\ + (\lambda_w + \lambda_l)\pi^2(b) - b] \end{aligned} \tag{11}$$

In equilibrium, $\mathrm{D}_b U^2(\pi^2(b),b) = 0$ which gives (5).

Conversely, let σ^2 solve (5). We show σ^2 is optimal. Suppose a bidder overbids to c so that $c > \sigma^2(t)$, i.e., $\pi^2(c) > t$. Then, from (11), we have

$$\begin{aligned} \mathrm{D}_c U^2(t,c) &= \mathrm{D}\pi^2(c) g(\pi^2(c),t)[(\lambda_w + \lambda_l + \alpha_R + \alpha_B - 1)t \\ &\quad + (\lambda_w + \lambda_l)\pi^2(c) - c] \\ &< \mathrm{D}\pi^2(c) g(\pi^2(c),t)[\Gamma(\alpha,\lambda)\pi^2(c) - c] \\ &= 0 \end{aligned}$$

Thus, overbid is not profitable. Similarly, it can be shown that underbid is also not profitable. So, σ^2 is optimal. ■

Proof of Proposition 5. From (3) and (5), we have

$$\begin{aligned}\sigma^1(t) &= \Gamma(\alpha,\lambda)\int_0^t \exp\Big\{-\int_y^t \frac{g(x,x)}{G(x,x)}\mathrm{d}x\Big\} y\frac{g(y,y)}{G(y,y)}\mathrm{d}y \\ &> \Gamma(\alpha,\lambda)t\frac{g(t,t)}{G(t,t)} \\ &> \Gamma(\alpha,\lambda)t \\ &= \sigma^2(t)\end{aligned}$$

■

Proof of Proposition 6. Consider bidder 1 with value t and bid b and consider bidder 2 who bids $\sigma_*^1(\mathcal{T})$. Bidder 1 wins if and only if $\mathcal{T} < \pi_*^1(b)$ and trade succeeds if and only if $\mathcal{T} > kt$. Therefore, with probability $\mathcal{T} < kt$, trade fails which gives bidder 1 a utility of $t - b$ and with probability $kt < \mathcal{T} < \pi_*^1(b)$, trade succeeds which gives him a utility of $(\alpha_R + \lambda_w)t + \lambda_l\mathcal{T} - b$.

Bidder 1 loses if and only if $\mathcal{T} > \pi_*^1(b)$ and trade succeeds if and only if $\mathcal{T} < t/k$. Therefore, with probability $\pi_*^1(b) < \mathcal{T} < t/k$, trade succeeds which gives bidder 1 a utility of $(1 - \alpha_B - \lambda_l)t - \lambda_w\mathcal{T} - b$ and with probability $\mathcal{T} > t/k$, trade fails which gives him a utility of $-b$.

Thus, the expected utility function can be written as (6). ■

Proof of Proposition 7. Let σ_*^1 be an equilibrium. Then, differentiating (6) w.r.t. b, we have

$$\begin{aligned}\mathrm{D}_bU_*^1(t,b) = \mathrm{D}\pi_*^1(b)g(\pi_*^1(c),t)[&(\alpha_R + \alpha_B + \lambda_w + \lambda_l - 1)t \\ &+ (\lambda_w + \lambda_l)\pi_*^1(b)] - 1\end{aligned} \tag{12}$$

In equilibrium, $\mathrm{D}_bU_*^1(\pi_*^1(b), b) = 0$ which gives

$$\mathrm{D}\pi_*^1(b) = \frac{1}{\Gamma(\alpha,\lambda)\pi_*^1(b)g(\pi_*^1(b),\pi_*^1(b))}$$

As $b = \sigma_*^1 \circ \pi_*^1(b)$, differentiating w.r.t. b gives $1 = \mathrm{D}\pi_*^1(b)\mathrm{D}\sigma_*^1 \circ \pi_*^1(b)$. Therefore, the above equation can be rewritten as

$$\mathrm{D}\sigma_*^1 \circ \pi_*^1(b) = \Gamma(\alpha,\lambda)\pi_*^1(b)g(\pi_*^1(b),\pi_*^1(b))$$

Implementing $t = \pi_*^1(b)$, we have

$$\mathrm{D}\sigma_*^1(t) = \Gamma(\alpha,\lambda)tg(t,t)$$

Using the fundamental theorem of calculus with $\sigma_*^1(0) = 0$, we have

$$\sigma_*^1(t) = \Gamma(\alpha,\lambda)\int_0^t xg(x,x)\mathrm{d}x$$

To show the converse, consider σ_*^1 that solves (7). Let a bidder with value t overbid to c where $c > \sigma_*^1(t)$, i.e., $t < \pi_*^1(c)$. Then, from (12), we have

$$\begin{aligned}\mathrm{D}_c U_*^1(t,c) = \mathrm{D}\pi_*^1(c) g(\pi_*^1(c), t)[(\alpha_R + \alpha_B + \lambda_w + \lambda_l - 1)t \\ + (\lambda_w + \lambda_l)\pi_*^1(c)] - 1\end{aligned}$$

Since $g(y, .)$ is strictly increasing for every $y \in \Re_+$, we have

$$\begin{aligned}\mathrm{D}_c U_*^1(t,c) &< \mathrm{D}\pi_*^1(c) g(\pi_*^1(c), \pi_*^1(c))[\Gamma(\alpha,\lambda)\pi_*^1(c)] - 1 \\ &= \mathrm{D}_c U_*^1(\pi_*^1(c), \pi_*^1(c)) \\ &= 0\end{aligned}$$

which implies that overbid is not profitable. Similarly, by reversing the inequalities, it can be shown that underbid is also not profitable. Therefore, π_*^1 is optimal. ■

Proof of Proposition 8. Consider bidder 1 with value t and bid b and consider bidder 2 who bids $\sigma_*^2(\mathcal{T})$. Bidder 1 wins if and only if $\mathcal{T} < \pi_*^2(b)$ and trade succeeds if and only if $\mathcal{T} > kt$. Therefore, with probability $\mathcal{T} < kt$, trade fails which gives bidder 1 a utility of $t - \sigma_*^2(\mathcal{T})$ and with probability $kt < \mathcal{T} < \pi_*^2(b)$, trade succeeds which gives him a utility of $(\alpha_R + \lambda_w)t + \lambda_l \mathcal{T} - \pi_*^2(\mathcal{T})$.

Bidder 1 loses if and only if $\mathcal{T} > \pi_*^2(b)$ and trade succeeds if and only if $\mathcal{T} < t/k$. Therefore, with probability $\pi_*^2(b) < \mathcal{T} < t/k$, trade succeeds which gives bidder 1 a utility of $(1 - \alpha_B - \lambda_l)t - \lambda_w \mathcal{T} - b$ and with probability $\mathcal{T} > t/k$, trade fails which gives him a utility of $-b$.

Thus, the expected utility function can be written as (8). ■

Proof of Proposition 9. Let σ_*^2 be an equilibrium. Then, differentiating (8), we have

$$\begin{aligned}\mathrm{D}_b U_*^2(t,b) = \mathrm{D}\pi_*^2(b) g(\pi_*^2(b), t)[(\alpha_R + \alpha_B + \lambda_w + \lambda_l - 1)t \\ + (\lambda_w + \lambda_l)\pi_*^2(b)] - 1 + G(\pi_*^2(b), t)\end{aligned} \tag{13}$$

The equilibrium condition $\mathrm{D}_b U_*^2(\pi_*^2(b), b) = 0$ gives

$$\mathrm{D}\pi_*^2(b) = \frac{1 - G(\pi_*^2(b), \pi_*^2(b))}{g(\pi_*^2(b), \pi_*^2(b))} \frac{1}{\Gamma(\alpha,\lambda)\pi_*^2(b)}$$

As $b = \sigma_*^2 \circ \pi_*^2(b)$ implies $1 = \mathrm{D}\pi_*^2(b)\mathrm{D}\sigma_*^1 \circ \pi_*^2(b)$, we have

$$\mathrm{D}\sigma_*^2 \circ \pi_*^2(b) = \frac{g(\pi_*^2(b), \pi_*^2(b))}{1 - G(\pi_*^2(b), \pi_*^2(b))}[\Gamma(\alpha,\lambda)\pi_*^2(b)]$$

Applying $t = \pi_*^2(b)$ gives

$$\mathrm{D}\sigma_*^2(t) = \frac{g(t,t)}{1-G(t,t)}[\Gamma(\alpha,\lambda)t] \tag{14}$$

Using the fundamental theorem of calculus with $\sigma_*^2(0) = 0$, we have

$$\sigma_*^2(t) = \Gamma(\alpha,\lambda)\int_0^t x\frac{g(x,x)}{1-G(x,x)}\mathrm{d}x$$

■

Proof of Proposition 10. From(7) and (9), the result follows. ■

Proof of Theorem 1. We show (1). Given $t \in (0,1)$ of a bidder, let $P^1(\sigma_*^1, t)$ be the interim expected payments of a bidder in the first-price all-pay auction and let $P^2(\sigma_*^2, t)$ be the interim expected payments of a bidder in the second-price all-pay auction. Then,

$$P^1(\sigma_*^1, t) = \sigma_*^1(t)$$

and

$$\begin{aligned} P^2(\sigma_*^2, t) &= \int_0^t \sigma_*^2(x)g(x,t)\mathrm{d}x + [1-G(t,t)]\sigma_*^2(t) \\ &= \Gamma(\alpha,\lambda)\int_0^t\int_0^x y\frac{g(y,y)}{1-F(y,y)}g(x,t)\mathrm{d}y\mathrm{d}x + [1-G(t,t)]\sigma_*^2(t) \end{aligned}$$

Using Fubini's theorem, we have

$$\begin{aligned} P^2(\sigma_*^2, t) &= \Gamma(\alpha,\lambda)\int_0^t\int_x^1 x\frac{g(x,x)}{1-G(x,x)}g(y,t)\mathrm{d}y\mathrm{d}x + [1-G(t,t)]\sigma_*^2(t) \\ &= \Gamma(\alpha,\lambda)\int_0^t x\frac{g(x,x)}{1-G(x,x)}[1-G(x,t)]\mathrm{d}x + [1-G(t,t)]\sigma_*^2(t) \end{aligned}$$

As $x < t$, from Property 5 of Lemma 1, we have

$$\begin{aligned} P^2(\sigma_*^2, t) &> \Gamma(\alpha,\lambda)\int_0^t xg(x,x)\mathrm{d}x + [1-G(t,t)]\sigma_*^2(t) \\ &= \sigma_*^1(t) + [1-G(t,t)]\sigma_*^2(t) \\ &= P^1(\sigma_*^1, t) + [1-G(t,t)]\sigma_*^2(t) \end{aligned}$$

So, $P^2(\sigma_*^2, t) > P^1(\sigma_*^1, t)$. Taking expectations, we have $R^2(\sigma_*^2) > R^1(\sigma_*^1)$.

We show (2). Given $t \in (0,1)$ of a bidder, let $P^1(\sigma^1, t)$ be the interim expected payments of a bidder in the first-price auction and let $P^1(\sigma_*^1, t)$

be the interim expected payments of a bidder in the first-price all-pay auction. Then,

$$\begin{aligned}
P^1(\sigma^1, t) &= G(t,t)\sigma^1(t) \\
&= \Gamma(\alpha, \lambda) G(t,t) \int_0^t \exp\Big\{ - \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x \Big\} y \frac{g(y,y)}{G(y,y)} \mathrm{d}y \\
&= \Gamma(\alpha, \lambda) \int_0^t y g(y,y) \frac{G(t,t)}{G(y,y)} \exp\Big\{ - \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x \Big\} \mathrm{d}y
\end{aligned}$$

Given $x > y$, from property 4 of Lemma 1, we have

$$- \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x < - \int_y^t \frac{g(x,y)}{G(x,y)} \mathrm{d}x$$

which implies

$$\begin{aligned}
- \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x &< - \int_y^t \mathrm{D} \ln G(x,y) \mathrm{d}x \\
&= \ln G(y,y) - \ln G(t,y)
\end{aligned}$$

Since $y < t$, from Property 5 of Lemma 1, we have

$$\begin{aligned}
- \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x &< \ln G(y,y) - \ln G(t,t) \\
&= \ln \frac{G(y,y)}{G(t,t)}
\end{aligned}$$

Applying the exponential function on both sides, we have

$$\exp\Big\{ - \int_y^t \frac{g(x,x)}{G(x,x)} \mathrm{d}x \Big\} < \frac{G(y,y)}{G(t,t)}$$

Using the above inequality in the expression of $P^1(\sigma^1, t)$, we have

$$\begin{aligned}
P^1(\sigma^1, t) &< \Gamma(\alpha, \lambda) \int_0^t x g(x,x) \mathrm{d}x \\
&= \sigma^1_*(t) \\
&= P^1(\sigma^1_*, t)
\end{aligned}$$

Thus, taking expectations, we have $R^1(\sigma^1_*) > R^1(\sigma^1)$.

We show (3). Given $t \in (0,1)$ of a bidder, let $P^1(\sigma^1, t)$ be the interim expected payments of a bidder in the first-price auction and let $P^2(\sigma^2, t)$ be the interim expected payments of a bidder in the second-price auction. Then,

$$P^2(\sigma^2_*, t) = \int_0^t \sigma^2_*(x) g(x,t) \mathrm{d}x + [1 - G(t,t)]\sigma^2_*(t)$$

Using integration-by-parts, we have

$$\begin{aligned}P^2(\sigma_*^2, t) &= \sigma_*^2(t)G(t,t) - \int_0^t \mathrm{D}\sigma_*^2(x)G(x,t)\mathrm{d}x + [1 - G(t,t)]\sigma_*^2(t) \\ &= \sigma_*^2(t) - \int_0^t \mathrm{D}\sigma_*^2(x)G(x,t)\mathrm{d}x\end{aligned}$$

Using (9) and (14), we have

$$\begin{aligned}P^2(\sigma_*^2, t) &= \Gamma(\alpha,\lambda)\int_0^t \frac{xg(x,x)}{1-G(x,x)}\mathrm{d}x - \Gamma(\alpha,\lambda)\int_0^t G(x,t)\frac{xg(x,x)}{1-G(x,x)}\mathrm{d}x \\ &= \Gamma(\alpha,\lambda)\int_0^t [1-G(x,t)]\frac{xg(x,x)}{1-G(x,x)}\mathrm{d}x \\ &= \Gamma(\alpha,\lambda)\int_0^t \frac{xg(x,x)}{1-G(x,x)}g(x,t)\frac{1-G(x,t)}{g(x,t)}\mathrm{d}x \\ &= \Gamma(\alpha,\lambda)\int_0^t xg(x,t)\frac{g(x,x)/[1-G(x,x)]}{g(x,t)/[1-G(x,t)]}\mathrm{d}x\end{aligned}$$

Given $t > x$ and Property 3 of Lemma 1, we have

$$\begin{aligned}P^2(\sigma_*^2, t) &> \Gamma(\alpha,\lambda)\int_0^t xg(x,t)\mathrm{d}x \\ &= P^2(\sigma^2, t)\end{aligned}$$

Thus, taking expectations, we have $R^2(\sigma_*^2) > R^2(\sigma^2)$.

We show (4). Given $t \in (0,1)$ of a bidder, let $P^1(\sigma^1, t)$ be the interim expected payments of a bidder in the first-price auction and let $P^2(\sigma^2, t)$ be the interim expected payments of a bidder in the second-price auction. Then, from the proof of Theorem 2, we have

$$P^1(\sigma^1, t) < \Gamma(\alpha,\lambda)\int_0^t xg(x,x)\mathrm{d}x \tag{15}$$

Given $x < t$ and the fact that $g(y,.)$ is strictly increasing for every $y \in \Re_+$, we have

$$\begin{aligned}P^1(\sigma^1, t) &< \Gamma(\alpha,\lambda)\int_0^t xg(x,x)\mathrm{d}x < \Gamma(\alpha,\lambda)\int_0^t xg(x,t)\mathrm{d}x \\ &= P^2(\sigma^2, t)\end{aligned} \tag{16}$$

Therefore, taking expectations, we have $R^2(\sigma^2) > R^1(\sigma^1)$.

We show (5). Given $t \in (0,1)$, from (15) and (16), we have

$$\begin{aligned}P^1(\sigma^1, t) &< \Gamma(\alpha,\lambda)\int_0^t xg(x,x)\mathrm{d}x = P^1(\sigma_*^1, t) \\ &< \Gamma(\alpha,\lambda)\int_0^t xg(x,t)\mathrm{d}x = P^2(\sigma^2, t)\end{aligned}$$

Therefore, taking expectations, we have $R^2(\sigma^2) > R^1(\sigma_*^1)$. ■

Proof of Theorem 2. We show (1). From Milgrom and Weber [14], we know

$$\beta^1(t) = \int_0^t \exp\Big\{ -\int_y^t \frac{g(x,x)}{G(x,x)}\mathrm{d}x \Big\} y \frac{g(y,y)}{G(y,y)}\mathrm{d}y$$

Comparing the above equation with (3), we have

$$\sigma^1(t) > \beta^1(t)$$

for every $t \in (0,1)$. The result follows immediately.

We show (2). From Milgrom and Weber [14], we know

$$\beta^2(t) = t$$

Comparing the above equation with (5), we have

$$\sigma^2(t) > \beta^2(t)$$

for every $t \in (0,1)$. The result follows immediately.

We show (3). From Krishna and Morgan [13], the equilibrium of the first-price all-pay auction without resale is characterized as

$$\beta^1_*(t) = \int_0^t x g(x,x)\mathrm{d}x$$

Comparing the above equation with (7), we have

$$\sigma^1_*(t) > \beta^1_*(t)$$

for every $t \in (0,1)$. The result follows immediately.

We show (4). From Krishna and Morgan [13], the equilibrium of the first-price all-pay auction without resale is characterized as

$$\beta^2_*(t) = \int_0^t x \frac{g(x,x)}{1-G(x,x)}\mathrm{d}x$$

Comparing the above equation with (9), we have

$$\sigma^2_*(t) > \beta^2_*(t)$$

for every $t \in (0,1)$. The result follows immediately. ■

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