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Arrowian social equilibrium: Indecisiveness, influence and rational social choices under majority rule

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Abstract

We introduce the concept of an Arrowian social equilibrium that inverts the schemata of Arrow (1950)'s famous impossibility theorem and captures the possibility of aggregating non-rational individual preferences into rational social preferences while respecting the Arrowian desiderata. Specifically, we consider individuals whose preferences may not be complete and who, accordingly, may be indecisive when faced with an issue. Breaking with tradition, we consider the possibility of such individuals drawing on their *beliefs* about society's preferences that result from the aggregation process to resolve their indecisiveness. Formally, individual choices are modeled as a rational shortlist method (Manzini and Mariotti, 2007), with own preferences followed by society's as the pair of ordered rationales. This results in a *mutual* interaction between individual and social choices. We study this interaction using majority rule as the aggregator, with an Arrowian social equilibrium specifying how individual and social choices are *co-determined*, while requiring the latter to be rational. Our main result identifies minimal levels of societal indecisiveness needed to guarantee the existence of such equilibrium.
JEL codes: D71, D91
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1 Introduction

In one of the most influential contributions to the social sciences, Arrow (1950) famously showed that it is not possible to aggregate every profile of rational individual preferences into a rational social preference if we require the aggregation rule to satisfy the Pareto principle (PP) and the conditions of independence of irrelevant alternatives (IIA) and no

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We have benefitted from conversations with Bhaskar Dutta, Arunava Sen and Debasis Mishra.

dictatorship (ND). In this paper, we re-visit this classic result. Our point of departure, drawing on behavioral insights, is the observation that individual preferences need not be rational in the classical sense assumed in Arrow's impossibility theorem. Allowing for such non-rational individual preferences, we propose a way in which social aggregation, while respecting the Arrowian conditions of PP, IIA and ND, results in rational social preferences.

To be more precise, we entertain the possibility that individual preferences may be incomplete. In that case when faced with an issue that is up for a vote, individuals may find themselves being indecisive when assessing this issue based on their preferences alone. If such indecisiveness is a real phenomenon—and we think it is—then a natural question presents itself: how do such voters arrive at a decision on which alternative to vote for? Our answer to this question is that such indecisive voters may be subject to social influence. Specifically, such voters may be influenced by their *beliefs* about society's preferences that emerge out of the preference aggregation process and they may use it to resolve their indecisiveness. For motivation, think of such voters as trusting the "wisdom of the crowd" or preferring to go with the winner when indecisive. The key theoretical possibility that this opens up is that of a *mutual* interaction between individual choices and social preferences. The individual to social direction is standard à la Arrow, but the opposite one of social preferences influencing individual choices is non-standard and the innovation introduced here. Of course, to meaningfully close down such mutual interactions, we have to propose an equilibrium notion. To that end, we introduce the notion of an *Arrowian social equilibrium*. Although this notion can be defined for any preference aggregator, to fix ideas better, we restrict attention to majority rule.[1]

An Arrowian social equilibrium under majority rule (ASEM) is a profile of choice functions, one for society and one each for the individuals in society, which are *mutually determined* in the following interactive way. On any issue, the social choice is determined by majority rule taking the individual choices on the issue as inputs. Majority rule social choice on any issue under an ASEM is required to be decisive and the resulting social choice function to be rational in the traditional sense of satisfying the weak axiom of revealed preference (WARP) and, hence, rationalized by a social preference ranking. When it comes to individual choices on an issue, for any such individual, if there is a unique maximal alternative according to her preferences, i.e., she is decisive on the issue, then that alternative is chosen. If not, the non-singleton set of maximal alternatives is assessed according to (her beliefs about) the social preference ranking and the best alternative according to it is chosen.[2] Formally, an individual's choices are determined according to the rational shortlist method of Manzini and Mariotti (2007), with her own preferences followed by society's being the ordered pair of rationales determining them.[3] The mutual

[1] Of course, majority rule satisfies the Arrowian conditions of PP, IIA and ND.

[2] As is standard, the equilibrium notion requires individuals to hold correct beliefs, in this case about the social preferences that result from the aggregation process.

[3] This particular way of modeling influence is similar to that in Cuhadaroglu (2017).

interactions between individual and social choices captured in an ASEM, therefore, invert the traditional Arrowian schemata. Whereas Arrow's impossibility theorem highlights social aggregation in the presence of PP, IIA and ND mapping rational individual behavior into non-rational social preferences, an ASEM, on the other hand, captures the opposite possibility of mapping non-rational individual behavior into rational social preferences.

To illustrate the concept of an ASEM using an example, consider the Condorcet voting paradox. Suppose three individuals, 1, 2 and 3, have the following preferences over the alternatives x, y and z: $x \succ_1 y \succ_1 z$, $y \succ_2 z \succ_2 x$, and $z \succ_3 x \succ_3 y$. In this case, as is well known, majority rule results in intransitive social preferences with a majority preferring x to y, a majority y to z, and a majority z to x. This means that for this preference profile an ASEM doesn't exist. Now suppose one of the individuals, say 1, is indecisive about her top alternative while the preferences of the other two individuals are the same; i.e., 1's preferences are now given by $\succ_1' = \{(x, z), (y, z)\}$, with $\succ_2$ and $\succ_3$ the same as above. With this minimal level of indecisiveness introduced to the preference profile, an ASEM indeed exists as illustrated in the table below, where c_1, c_2 and c_3 denote the three individual's choice functions, and c_0 the social choice function.

	$\{x, y\}$	$\{y, z\}$	$\{x, z\}$	$\{x, y, z\}$
c_1	$\boxed{y}$	y	x	$\boxed{y}$
c_2	y	y	z	y
c_3	x	z	z	z
c_0	y	y	z	y

Observe that given the individual choices, the social choice on every issue is the majority winner. Further, the social choice function satisfies WARP and is rationalized by the preference ranking $y \succ_0 z \succ_0 x$. Coming to individual behavior, individuals 2 and 3 are rational and on any issue, they have a unique maximal alternative according to their preferences that is chosen. This is also the case for individual 1 on issues $\{y, z\}$ and $\{x, z\}$ but not on issues $\{x, y\}$ and $\{x, y, z\}$. On these two issues, the set of maximal or undominated alternatives according to her preferences is given by $\{x, y\}$. Under an ASEM, faced with such indecisiveness, an individual resolves it based on correct beliefs about society's preferences. Since society prefers y to x, therefore, 1's choice on both these issues is y. In other words, for this preference profile, the interactions between individual and social choices underlying an ASEM are mutually consistent.

Our key theoretical enquiry is regarding the existence of ASEM and its connection to societal indecisiveness. Of course, if the preferences of all individuals are rational, in particular complete, and there is no indecisiveness in society, following Arrow's theorem, an ASEM is not guaranteed to exist, as we saw above for the original Condorcet preferences. Therefore, the question of existence boils down to the presence of a sufficient level of indecisiveness in society. To systematically pursue the question, we construct an index of average societal indecisiveness that takes a value between 0 (no indecisiveness) and 1 (complete indecisiveness) on any issue and is based on the size of individuals' preference maximal sets. Our main result identifies the minimal levels of societal indecisiveness across issues (as measured by this index) that must be there to guarantee existence of ASEM.

Our work relates to approaches to the Arrowian aggregation problem in which individual preferences are not required to be rational in the classical sense and individual behavior, as inputs to the aggregation process, is allowed to violate WARP. For a discussion of such work, the reader may refer to Aizerman and Aleskerov (1986), Aizerman and Aleskerov (1995), Aleskerov (2002) and Aleskerov (2013), amongst others. It is known that successful aggregation satisfying Arrow's desiderata is possible on such domains (Aizerman and Aleskerov, 1986). In recent times, Katz and Sandroni (2020) and Sandroni and Sandroni (2021) have contributed to this area. The former work with the weak WARP domain and show that a delegation rule that assigns a social decider to each issue satisfies Arrow's desiderata. The latter show that it is possible to aggregate individual choice functions that satisfy almost any condition weaker than WARP into a social choice function that satisfies the same condition and Arrow's desiderata. The main difference between our work and this literature is two-folds. First, we require the output of the aggregation process, i.e., social choices, to be rational. Second, we allow the inputs to the aggregation process, i.e., individual choices, to be endogenously determined, in particular, to be influenced by the output of the aggregation process itself.

The rest of the paper is organized as follows. The next section lays out the primitives. Section 3 formally defines an ASEM, following which Section 4 presents our main result that provides a sufficient condition for existence of ASEM in terms of societal indecisiveness. Proofs of results appear in the Appendix.

2 Primitives

Preferences and indecisiveness. We consider a society with n individuals, with $I = \{1, \ldots, n\}$ denoting the set of individuals. X denotes a finite set of alternatives, with typical elements denoted by x, y etc. We assume that X has at least 3 alternatives. $\mathcal{X}$ denotes the set of non-empty, non-singleton subsets of X with typical elements denoted by S, T etc. We refer to any element of $\mathcal{X}$ as an issue. Each individual $i \in I$ has preferences over the set of alternatives captured by an asymmetric and acyclic binary relation $\succ_i \subseteq X \times X$. Denote by $\mathcal{B}$ the set of all asymmetric and acyclic binary relations. The fact that $\succ_i$ need not be complete means that given any issue $S \in \mathcal{X}$, i may not be able to come up with a decisive choice based on $\succ_i$. Formally, for any asymmetric and acyclic binary relation $B \in \mathcal{B}$ and any issue $S \in \mathcal{X}$, denote the set of B-maximal alternatives of

S by:

$$m(S;B) = \{x \in S : \nexists y \in S \text{ s.t. } (y,x) \in B\}$$

Given that $\succ_i$ is acyclic, for any issue S, $m(S;\succ_i)$ is non-empty. But, given that the relation may not be complete, $m(S;\succ_i)$ need not be a singleton. As such, any of the individuals in society, faced with an issue, may experience indecisiveness when evaluating that issue based on her preferences alone. The fact that such indecisiveness may exist is a key ingredient of our model. For any issue S, we can think of an index of individual indecisiveness in terms of the size of the set $m(S;\succ_i)$; the larger the number of alternatives in this set, greater is the extent of indecisiveness that the individual experiences. But, to get an appropriate measure of indecisiveness that is comparable across issues, we would need to normalize by the size of the issue. So, for any $\succ_i \in \mathcal{B}$, define an index of individual indecisiveness on the issue S by:[4]

$$\theta_{\succ_i}(S) = \frac{|m(S;\succ_i)| - 1}{|S| - 1}$$

If the restriction of $\succ_i$ to S, denote it by $\succ_{i,S}$, is complete, then $m(S;\succ_i)$ is a singleton and $\theta_{\succ_i}(S) = 0$. On the other hand, if $\succ_{i,S} = \emptyset$, then $m(S;\succ_i) = S$ and $\theta_{\succ_i}(S) = 1$. Hence this index takes values between 0 and 1 with higher values signifying greater indecisiveness. Drawing on this, for any profile of individual preferences, $\succ = (\succ_1, \ldots, \succ_n) \in \mathcal{B}^n$, we can define a measure of average societal indecisiveness on the issue S by:

$$\theta_{\succ}(S) = \frac{1}{n}\sum_{i=1}^{n} \theta_{\succ_i}(S)$$

Choice and social choice. A choice function $c : \mathcal{X} \to X$ is a mapping that, for any issue $S \in \mathcal{X}$, specifies the alternative $c(S) \in S$ that is chosen in that issue. Denote the set of all choice functions by $\mathcal{C}$. We refer to a choice function as rational if it satisfies WARP. Recall that WARP imposes the following consistency on choices: for all $S, T \in \mathcal{X}$ with $S \subseteq T$, if $c(T) \in S$, then $c(S) = c(T)$. That is, removing some of the unchosen alternatives from a menu should not change the choice.[5] If a choice function satisfies WARP, then it can be rationalized by a strict preference ranking[6] that can be uniquely elicited from it. Specifically, define the base relation P of the choice function c by xPy if $x = c(\{x,y\})$, $x, y \in X$, $x \neq y$. If c satisfies WARP, then P is a strict preference ranking and for any $S \in \mathcal{X}$, $c(S) = m(S;P)$.

In this paper, we focus on aggregating individual choices to a social choice based on majority rule. Given any issue S and choices of each of the individuals on that issue,

[4] $|.|$ denotes the cardinality of a set.

[5] WARP can be equivalently stated as follows: for all $S, T \in \mathcal{X}$ and $x, y \in S \cap T$, $x \neq y$, if $c(S) = x$ then $c(T) \neq y$. That is, if x is chosen in the presence of y in some issue, then y is not chosen in any issue in which x is present.

[6] By a strict preference ranking, we mean a binary relation that is total, asymmetric and transitive.

$(c_1(S), \ldots, c_n(S))$, the majority rule (MR) choice for society is given by:

$$f_{mr}(c_1(S), \ldots, c_n(S)) = \begin{cases} x, & \text{if } x \text{ s.t. } |\{i \in I : c_i(S) = x\}| > \frac{n}{2} \\ \emptyset, & \text{otherwise} \end{cases}$$

In other words, if more than 50% of individuals choose an alternative in an issue, then it is society's choice based on MR. On the other hand, if such a level of support for an alternative is not there in an issue, then MR is indecisive and cannot make a choice for society. Therefore, in relation to the Arrowian desiderata, although MR satisfies PP, IIA and ND, it doesn't satisfy the requirement of non-empty social choice. Further, even if social choices under MR are non-empty, they need not be rational.

Rational shortlist method. The final ingredient that we need to set up our model is a sequential choice procedure that was introduced by Manzini and Mariotti (2007), called the rational shortlist method (RSM). A choice function $c : \mathcal{X} \to X$ is an RSM if there exists an ordered pair of asymmetric binary relations $(P_1, P_2) \in \mathcal{B} \times \mathcal{B}$ such that for all $S \in \mathcal{X}$,

$$c(S) = m(m(S; P_1); P_2)$$

In other words, choice from any issue S is made via a two stage sequential process. First, the binary relation P_1 is used to shortlist the set of undominated alternatives in S according to it, $m(S; P_1)$. Then the binary relation P_2 is used to make the choice from this shortlisted set, i.e., after the two stages a decisive choice must result. Of course, if $m(S; P_1)$ is a singleton then the second stage is redundant.

3 Arrowian social equilibrium under majority rule

We can now define the concept of an Arrowian social equilibrium introduced in this paper. The concept captures how individual behavior, specified in the definition below by the choice functions $(c_1, \ldots, c_n) \in \mathcal{C}^n$, and social choice, specified by the choice function $c_0 \in \mathcal{C}$, mutually interact and are co-determined.

Definition 3.1. *Given $\succ = (\succ_1, \ldots, \succ_n) \in \mathcal{B}^n$, a collection $(c_0, c_1, \ldots, c_n) \in \mathcal{C}^{n+1}$ is an Arrowian social equilibrium under majority rule (ASEM) if for all $S \in \mathcal{X}$,*

1. *$c_0(S) = f_{mr}(c_1(S), \ldots, c_n(S))$ and c_0 satisfies WARP; and*
2. *$c_i(S) = m(m(S; \succ_i); \succ_0)$, $\forall i \in I$, where $\succ_0 \subseteq X \times X$ is defined by $x \succ_0 y$ if $x = c_0(\{x, y\})$*

Under an ASEM, the choice function c_0 captures social choices and is determined by MR. Given that $c_0 \in \mathcal{C}$ and is required to satisfy WARP, an ASEM demands that not only should society be able to choose in any issue based on MR but also that such social choices must be rational, i.e., they are rationalized by the ranking $\succ_0$, the base relation of c_0, given by: $x \succ_0 y$ if $x = c_0(\{x, y\})$. In other words, an ASEM imposes Arrow's original requirement of rational social choices. The "non-standard" feature of the definition is the specification of individual choices, which are not taken as exogenous to the aggregation process but rather are endogenous as they can be influenced by society's preferences whenever individual preferences result in indecisiveness. Specifically, choices of any individual $i \in I$ are determined according to an RSM based on the ordered pair $(\succ_i, \succ_0)$. That is, faced with any issue S, individual i first assesses that issue according to her preferences $\succ_i$ and shortlists the maximal set, $m(S; \succ_i)$. If this set is a singleton, then she chooses the maximal alternative. If not, she evaluates the shortlisted alternatives according to (her beliefs about) society's preferences, $\succ_0$. Observe that, like in any equilibrium notion, in an ASEM too, individuals are required to hold correct beliefs, in this case about society's preferences, $\succ_0$, that result from the aggregation process.

We now present a couple of examples, the first illustrating existence of an ASEM and the second non-existence. Note that in the first example we abuse notation by suppressing set delimiters; e.g., we write xyz instead of $\{x, y, z\}$, $c(xy)$ instead of $c(\{x, y\})$, etc.

Example 3.1 (ASEM exists)**.** Consider a society with 5 individuals and 4 alternatives, $I = \{1, 2, 3, 4, 5\}$ and $X = \{x, y, z, w\}$. The preferences of the five individuals are as follows:

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$
$\{(y,z),(y,w),(z,w)\}$	$\{(x,w),(x,z),(w,z)\}$	$\{(w,y),(w,x),(y,x)\}$	$\{(z,x),(z,y),(x,y)\}$	$\{(z,y),(z,w),(y,w)\}$

Given these individual preferences, the table below specifies for all issues, the set of $\succ_i$-maximal alternatives, $m(\,.\,; \succ_i), i \in I$, and a collection $(c_0, c_1, c_2, c_3, c_4, c_5)$ which forms an ASEM.

		xy	xz	xw	yz	yw	zw	xyz	xyw	xzw	yzw	$xyzw$
1	$m(S;\succ_1)$	xy	xz	xw	y	y	z	xy	xy	xz	y	xy
	$c_1(S)$	y	z	w	y	y	z	y	y	z	y	y
2	$m(S;\succ_2)$	xy	x	x	yz	yw	w	xy	xy	x	yw	xy
	$c_2(S)$	y	x	x	z	y	w	y	y	x	y	y
3	$m(S;\succ_3)$	y	xz	w	yz	w	zw	yz	w	zw	zw	zw
	$c_3(S)$	y	z	w	z	w	z	z	w	z	z	z
4	$m(S;\succ_4)$	x	z	xw	z	yw	zw	z	xw	zw	zw	zw
	$c_4(S)$	x	z	w	z	y	z	z	w	z	z	z
5	$m(S;\succ_5)$	xy	xz	xw	z	y	z	xz	xy	xz	z	xz
	$c_5(S)$	y	z	w	z	y	z	z	y	z	z	z
	$c_0(S)$	y	z	w	z	y	z	z	y	z	z	z

It is straightforward to see that c_0 satisfies WARP and is rationalized by the ranking $\succ_0$ given by $z \succ_0 y \succ_0 w \succ_0 x$. Further, given $(c_1, \ldots, c_5)$, it can be verified that, in any issue S, the social choice $c_0(S)$ is determined by MR. In addition, for each $i \in I$, c_i is an RSM based on the ordered pair $(\succ_i, \succ_0)$. E.g., consider individual 1 and the issue $\{y, z, w\}$. Since $m(yzw; \succ_1)$ is a singleton, the choice is determined in the first stage, based solely on her preferences, i.e., $c_1(yzw) = m(yzw; \succ_1) = y$. Now, consider the issue $\{x, y, z\}$. Since $m(xyz; \succ_1) = \{x, y\}$, the choice is determined by society's preference, $\succ_0$, in the second stage. Specifically, $c_1(xyz) = m(m(xyz; \succ_1); \succ_0) = m(xy; \succ_0)$, i.e., $c_1(xyz) = y$, since $y \succ_0 x$.

Example 3.2 (ASEM does not exist)**.** Consider a society with 5 individuals and 4 alternatives, $I = \{1, 2, 3, 4, 5\}$ and $X = \{x, y, z, w\}$. The preferences of individuals and the resulting maximal sets on the issue X are as follows:

$$
\begin{aligned}
\succ_1 &= \{(x,z),(x,w),(w,y)\} & & & m(X;\succ_1) &= \{x\} \\
\succ_2 &= \{(y,x),(z,y)\} & & & m(X;\succ_2) &= \{z,w\} \\
\succ_3 &= \{(y,z),(y,w)\} & &\Longrightarrow & m(X;\succ_3) &= \{x,y\} \\
\succ_4 &= \{(w,x),(w,z)\} & & & m(X;\succ_4) &= \{y,w\} \\
\succ_5 &= \{(x,y),(z,x),(z,w)\} & & & m(X;\succ_5) &= \{z\}
\end{aligned}
$$

As can be verified, for the issue X, there is no alternative that is in a majority of the $m(X; \succ_i)$ sets, i.e., for all $x \in S$, $|\{i \in I : x \in m(X; \succ_i)\}| \leq 2$. Since $c_i(S) \in m(X; \succ_i)$ under an ASEM, no MR winner can exist in this issue.

4 Indecisiveness and existence of ASEM

For any preference profile $\succ = (\succ_1, \ldots, \succ_n)$, if all individual preferences are "fully incomplete," i.e., $\succ_i = \emptyset$, for all $i \in I$, then the index of average societal indecisiveness,

$$\theta_{\succ}(S) = \frac{1}{n} \sum_{i=1}^{n} \theta_{\succ_i}(S) = \frac{1}{n} \sum_{i=1}^{n} \frac{|m(S; \succ_i)| - 1}{|S| - 1},$$

is equal to 1 for all issues. In that case an ASEM trivially exists. On the other hand, if we restrict attention to those preferences profiles in which all individual preferences are complete, then $\theta_{\succ}(S) = 0$, for all issues. In that case, an ASEM is not guaranteed to exist. Therefore, the key analytical exercise when it comes to the existence of ASEM is to identify minimal levels of societal indecisiveness that's needed to guarantee existence. We now turn to that exercise. In the following proposition we identify indecisiveness bounds such that for all preference profiles in which average societal indecisiveness exceeds these bounds, existence is guaranteed. We first present the result for the case where the number of individuals, n, is odd. For the case of n even, a minor additional condition is needed (essentially for dealing with a tie-breaking issue in the proof), which we discuss in a remark below. In the way of notation, for any $r \in \mathbb{R}_+$, $[r]$ will denote the greatest integer less than or equal to r.

Proposition 4.1 (n odd)**.** *If* $\succ = (\succ_1, \ldots, \succ_n) \in \mathcal{B}^n$ *such that*

$$\theta_\succ(S) > \underline{\theta}(S) := \frac{\frac{|S|}{n}\left[\frac{n}{2}\right] - 1}{|S| - 1}, \text{ for all } S \in \mathcal{X} \text{ with } |S| > 2,$$

an ASEM exists.

Proof: Please refer to Section A.1.

The Proposition helps us identify a sub-domain of $\mathcal{B}^n$ based on our average societal indecisiveness index such that for any profile from that sub-domain, an ASEM is guaranteed to exist. For any issue S, the indecisiveness bound $\underline{\theta}(S)$ depends on the size of the issue and the number of individuals in society.[7] Further, it is straightforward to verify that $\underline{\theta}(.)$ is increasing in the size of the issue and is bounded above by $\frac{1}{2}$. To illustrate the result, re-visit Example 3.1. In that example $n = 5$. Hence, the indecisiveness bounds are given by:

$$\underline{\theta}(S) = \begin{cases} \frac{\frac{3}{5}\left[\frac{5}{2}\right]-1}{|3|-1} = \frac{\frac{3.2}{5}-1}{2} = \frac{1}{10}, & \text{if } |S| = 3 \\ \\ \frac{\frac{4}{5}\left[\frac{5}{2}\right]-1}{|4|-1} = \frac{\frac{4.2}{5}-1}{3} = \frac{1}{5}, & \text{if } |S| = 4 \end{cases}$$

On the other hand, the average societal indecisiveness indices for the different menus of size three and four are as shown below. As we can see, for all such menus, $\theta_\succ(S) > \underline{\theta}(S)$.

	$S = xyz$		$S = xyw$		$S = xzw$		$S = yzw$		$S = xyzw$	
	$m(S;\succ_i)$	$\theta_{\succ_i}(S)$	$m(S;\succ_i)$	$\theta_{\succ_i}(S)$	$m(S;\succ_i)$	$\theta_{\succ_i}(S)$	$m(S;\succ_i)$	$\theta_{\succ_i}(S)$	$m(S;\succ_i)$	$\theta_{\succ_i}(S)$
1	xy	$\frac{1}{2}$	xy	$\frac{1}{2}$	xz	$\frac{1}{2}$	y	0	xy	$\frac{1}{3}$
2	xy	$\frac{1}{2}$	xy	$\frac{1}{2}$	x	0	yw	$\frac{1}{2}$	xy	$\frac{1}{3}$
3	yz	$\frac{1}{2}$	w	0	zw	$\frac{1}{2}$	zw	$\frac{1}{2}$	zw	$\frac{1}{3}$
4	z	0	xw	$\frac{1}{2}$	zw	$\frac{1}{2}$	zw	$\frac{1}{2}$	zw	$\frac{1}{3}$
5	xz	$\frac{1}{2}$	xy	$\frac{1}{2}$	xz	$\frac{1}{2}$	z	0	xz	$\frac{1}{3}$
$\theta_\succ(S)$		$\frac{2}{5}$		$\frac{2}{5}$		$\frac{2}{5}$		$\frac{3}{10}$		$\frac{1}{3}$

Tightness of the bound. Example 3.2, where an ASEM doesn't exist, can be used to show that the indecisiveness bounds identified in Proposition 4.1 are tight. For $|S| = 4$

[7] The dependence of $\underline{\theta}(S)$ on n is so for the case of n odd only. As we will see below, it is not so when n is even.

and $n = 5$, we have already calculated that $\underline{\theta}(S) = \frac{1}{5}$. For the issue $X = \{x, y, z, w\}$, we can verify from above that $\sum_{i=1}^{5} m(X; \succ_i) = 8$. Accordingly,

$$\theta_{\succ}(X) = \frac{1}{n} \sum_{i=1}^{n} \frac{|m(X; \succ_i)| - 1}{|S| - 1} = \frac{1}{5} \frac{\sum_{i=1}^{5} |m(X; \succ_i)| - 5}{3} = \frac{1}{5} \times \frac{8 - 5}{3} = \frac{1}{5}$$

That is $\theta_{\succ}(X)$ is exactly equal to $\underline{\theta}(S) = \frac{1}{5}$ and not greater than it; and we have non-existence of ASEM. This shows that the bound is tight.

The case of n even. For the case of n even, we need an additional condition to establish that the indecisiveness bounds identified in Proposition 4.1 guarantees existence of ASEM. The condition requires that in any binary issue, there exists at least one individual in society who is indecisive over this issue.

Proposition 4.2 (n even)**.** *If $\succ = (\succ_1, \ldots, \succ_n) \in \mathcal{B}^n$ such that*

$$\theta_{\succ}(S) > \underline{\theta}(S) = \frac{\frac{|S|}{n}\left[\frac{n}{2}\right] - 1}{|S| - 1} = \frac{1}{2}\left(\frac{|S| - 2}{|S| - 1}\right), \text{ for all } S \in \mathcal{X} \text{ with } |S| > 2,$$

and for all $x, y \in X$, there exists $i \in I$ such that neither $x \succ_i y$ nor $y \succ_i x$, then an ASEM exists.

Proof: Please refer to Section A.2.

It is worth pointing out that, unlike in the case of n odd, in the case of n even, the bound $\underline{\theta}(S)$ does not depend on the number of individuals in society.

A final remark. It should be straightforward to see that an Arrowian social equilibrium (ASE) can be defined for any social aggregator, specifically ones that satisfy the Arrowian desiderata of PP, IIA and ND. For any such aggregator an ASE trivially exists when the indecisiveness index is 1 for all issues (individual preferences are fully incomplete); and it is not guaranteed to exist when this index is 0 for all issues (individual preferences are complete). An open question with respect to such aggregators is about the minimal indecisiveness required to generate rational social preferences under an ASE. We leave it to future work to throw light on this question.

A Appendix

A.1 Proof of Proposition 4.1

Consider any preference profile $\succ = (\succ_1, \ldots, \succ_n) \in \mathcal{B}^n$ for which $\theta_{\succ}(S) > \underline{\theta}(S) = \frac{\frac{|S|}{n}\left[\frac{n}{2}\right] - 1}{|S| - 1}$, for all $S \in \mathcal{X}$ with $|S| > 2$. For any such profile, we show below that an ASEM exists by explicitly constructing one.

Step 1: Showing that MR candidates exist for any issue.

For any issue $S \in \mathcal{X}$ with $|S| > 2$, first note that the set of potential MR winners under an ASEM is given by:

$$M(S) = \left\{x \in S : |\{i \in I : x \in m(S; \succ_i)\}| > \frac{n}{2}\right\} = \left\{x \in S : |\{i \in I : x \in m(S; \succ_i)\}| > \left[\frac{n}{2}\right]\right\}$$

This is because in any ASEM, for any such S, $c_i(S) \in m(S, \succ_i)$, and hence for an alternative to be chosen by a majority, a necessary condition is that it is in a majority of individual maximal sets. We first establish that if $\theta_{\succ}(S) > \underline{\theta}(S)$, then $M(S) \neq \emptyset$, for any such S. To do so assume otherwise; suppose $M(S) = \emptyset$, for some such S. For any $x \in S$, let $I_S(x) = \{i \in I : x \in m(S; \succ_i)\}$. Under our supposition $|I_S(x)| \leq [\frac{n}{2}]$, for all $x \in S$. It should be straightforward to verify that:

$$\sum_{i=1}^{n} |m(S; \succ_i)| = \sum_{x \in S} |I_S(x)|$$

This implies that:

$$\sum_{i=1}^{n} |m(S; \succ_i)| = \sum_{x \in S} |I_S(x)| \leq |S| \left[\frac{n}{2}\right] \implies \sum_{i=1}^{n} |m(S; \succ_i)| - n \leq |S| \left[\frac{n}{2}\right] - n$$

$$\implies \frac{1}{n} \sum_{i=1}^{n} \Big(|m(S; \succ_i)| - 1\Big) \leq \frac{|S|}{n} \left[\frac{n}{2}\right] - 1 \implies \frac{1}{n} \sum_{i=1}^{n} \frac{|m(S; \succ_i)| - 1}{|S| - 1} \leq \frac{\frac{|S|}{n}\left[\frac{n}{2}\right] - 1}{|S| - 1}$$

i.e., $\theta_{\succ}(S) \leq \underline{\theta}(S)$, which brings us to our desired contradiction.

Step 2: Construction of ASEM choice profile $(c_0, c_1, \ldots, c_n) \in \mathcal{C}^{n+1}$

In this step, we explicitly construct a choice profile $(c_0, c_1, \ldots, c_n) \in \mathcal{C}^{n+1}$, and show that it constitutes an ASEM. To that end, first, observe the following: if $S, T \in \mathcal{X}$ such that $S \subseteq T$, then for any $i \in I$, $m(T; \succ_i) \cap S \subseteq m(S; \succ_i)$. This is because if for any $x \in S$, $\nexists y \in T$, such that $y \succ_i x$, then clearly $\nexists y \in S \subseteq T$, such that $y \succ_i x$. In turn, this implies that $M(T) \cap S \subseteq M(S)$.

For notational convenience, in the construction below, write $X = \{x_1, \ldots, x_k\}$. Since, from Step 1, $M(S) \neq \emptyset$ for all $S \in \mathcal{X}$, $|S| > 2$, we can construct $c_0 : \mathcal{X} \to X$ recursively as follows:

Step 2.1: Wlog, let $x_1 \in M(X)$.
Let $c_0(S) = x_1$, for all $S \in \mathcal{X}$ such that $x_1 \in S$.

Step 2.2: Wlog, let $x_2 \in M(X \setminus \{x_1\})$.

Let $c_0(S) = x_2$, for all $S \in \mathcal{X}$, such that, $x_2 \in S$ and $x_1 \notin S$.

$$\vdots$$

Step 2.k-2: Wlog, let $x_{k-2} \in M(X \setminus \{x_1, x_2, \dots, x_{k-3}\}) = M(\{x_k, x_{k-1}, x_{k-2}\})$.
Let $c_0(S) = x_{k-2}$, for all $S \in \mathcal{X}$ such that $x_{k-2} \in S$ and $x_1, \dots, x_{k-3} \notin S$.

Observe that in this manner, we have covered all issues except for the issue $\{x_k, x_{k-1}\}$. Since n is odd, even if all individuals have complete preferences over x_k and x_{k-1}, we have, $M(\{x_k, x_{k-1}\}) \neq \emptyset$. Wlog say, $x_{k-1} \in M(\{x_k, x_{k-1}\})$. Let $c_0(\{x_k, x_{k-1}\}) = x_{k-1}$. It is straightforward to see that c_0 constructed thus is a choice function. It also directly follows from the construction that c_0 satisfies WARP. Let $\succ_0 \subseteq X \times X$ be defined by $x \succ_0 y$ if $c_0(\{x, y\}) = x$. Since, c_0 satisfies WARP, it follows that $\succ_0$ is a strict preference ranking and it rationalizes c_0, i.e., for all $S \in \mathcal{X}$, $c_0(S) = m(S; \succ_0)$. Finally, note that in this construction, drawing on the fact that if $T' \subseteq T$, then $M(T) \cap T' \subseteq M(T')$, we have that for all $S \in \mathcal{X}$, if $c_0(S) = x$, then $x \in M(S)$.

Next, we construct $c_1, c_2, \dots, c_n$ as follows. For any $S \in \mathcal{X}$ and $i \in I$, let:

$$c_i(S) = m(m(S; \succ_i); \succ_0)$$

Since $\succ_0$ is a strict preference ranking, the second stage is always decisive and, hence, any such c_i is a well defined choice function (specifically, an RSM).

All that, therefore, remains to be shown to establish that the collection $(c_0, c_1, \dots, c_n)$ is an ASEM is that for all $S \in \mathcal{X}$, $c_0(S) = f_{mr}(c_1(S), \dots, c_n(S))$. Consider any $S \in \mathcal{X}$. Let $c_0(S) = x$. Then it follows that $x \succ_0 y$, for all $y \in S \setminus \{x\}$. Accordingly, for any $i \in I$, if $x \in m(S; \succ_i)$, then $c_i(S) = x$. This is, of course, true if $m(S; \succ_i) = \{x\}$. Even otherwise, this is so because x is the best alternative in $m(S; \succ_i)$ according to $\succ_0$. Further, we know that $x \in M(S)$. That is,

$$\begin{aligned} |\{i : x \in m(S; \succ_i)\}| > \frac{n}{2} \quad &\implies \quad |\{i : x = c_i(S)\}| > \frac{n}{2} \\ &\implies \quad x = f_{mr}(c_1(S), \dots, c_n(S)), \text{ as required.} \end{aligned}$$

A.2 Proof of Proposition 4.2

The proof for the case n even is exactly along the same lines as n odd. The only difference emerges in Step 2, following the $(k-2)$-th step when we arrive at the issue $\{x_k, x_{k-1}\}$. Now, it is possible that all individuals have complete preferences over these two alternatives and exactly half prefer x_k to x_{k-1} and the other half x_{k-1} to x_k. In that case $M(\{x_k, x_{k-1}\}) = \emptyset$. This is precisely where the role of the additional condition comes in. According to it, there exists some $i \in I$ such that neither $x_k \succ_i x_{k-1}$, nor $x_{k-1} \succ_i x_k$. This implies $M(\{x_k, x_{k-1}\}) \neq \emptyset$ and the rest of the argument follows as above.

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