Ashoka University
Economics Discussion Paper 168

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August 2026

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August 27, 2026

Abstract

Consider a finite set of potential bidders for the sale of an indivisible object where every bidder has an outside option. Bidders have private information about the object's value and their outside option which are drawn from two different probability distributions. Every bidder participates in the auction if and only if their value for the object is larger than their outside option, which leads to uncertain number of participants. The seller imposes a floor on the number of participating bidders in order to conduct an auction. If the floor is not met, the auction is canceled. We show that if either the value distribution is strengthened or the outside option distribution is weakened, the bidders raise their bids. We also show that the bidders raise their bids due to the imposition of a floor.

JEL classification: D44, D82

Keywords: outside option, stochastic bidders, stochastic floor, first-price auction, bid behavior

1 Introduction

Almost every bidder participating in an auction has an outside option that is privately known to him. If a bidder's outside option is worse than the object that is being sold in the auction, the bidder participates in the auction. Otherwise he does not participate. As a result, the number of bidders participating in the auction becomes uncertain.

In certain auctions, a floor on the number of bidders is imposed by the seller in order to conduct an auction due to the following reasons: (a) strict regulation by a governing authority so as to minimize favoritism, (b) boost competition that raises efficiency and revenues, (c) seller's reserve prices are sufficiently high, (d) discourage collusion among bidders and avoid bid rigging. Usually, a floor on the number of bidders is hidden

*I thank Claude and Refine for useful research assistance.

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due to the following reasons: (a) to avoid manipulation of participation, (b) to protect the seller's bargaining power, (c) to protect the object's future price and the seller's reputation.

Consider a sealed-bid auction for an indivisible object with a finite number of potential bidders. Every bidder has an outside option that is substitutable to the object sold in the auction. The values of the object and the outside option are private information for every bidder. Bidders draw a two-dimensional signal from two statistically independent probability distributions where the first signal indicates their value of the object and the second signal indicates their value of the outside option. Every bidder participates if and only if their object's value is higher than their outside option. The seller imposes a floor on the number of bidders in order to conduct the auction. If the floor is not met, the seller cancels the auction. The floor is private information to the seller and the bidders only observe a common prior on the floor. Thus, outside option leads to uncertainty about the number of participants while a floor leads to uncertainty about the occurrence of the auction.

Although, a priori, the signals of the object's value and its outside option are drawn from independent probability distributions, there is interdependence between the object's values and the outside option for the participating bidders because their value of the object is greater than the outside option. Therefore, a bidder's belief about a participating bidder will be defined on a triangle and not a rectangle.

In this paper, we examine the following questions:

1. How is an equilibrium characterized under uncertain floor, certain floor and no floor on the number of bidders?
2. How does the heterogeneity between the value distribution and outside option distribution impact the bids?
3. What is the impact on bids due to an imposition of a floor on the number of bidders?

In Theorems 1-2 and Corollaries 1-4, we provide certain characterizations of an equilibrium where the bid function depends on a two-dimensional signal. Theorem 1 characterizes an equilibrium when the probability distributions of the object and the outside option are asymmetric across the bidders and the floor on the number of participants is uncertain. Theorem 2 characterizes a symmetric equilibrium when the floor is uncertain. In Corollary 1, we consider that the floor is certain and characterize an equilibrium with asymmetric bidders. Corollary 3 is the counterpart of Corollary 1 for symmetric bidders. In Corollary 4, we characterize an asymmetric equilibrium when there is no floor. Corollary 5 is the counterpart of Corollary 4 for symmetric bidders.

The impact of heterogeneity between the value distribution and outside option distribution is captured by fixing either the probability distribution of the object's value or the probability distribution of the outside option and perturbing the other probability distribution. Theorem 3 shows that if the probability distribution of the object's value is strengthened relative to the probability distribution of the outside option, then the bidders start behaving aggressively. Theorem 4 shows that if the probability distribution of the outside option is strengthened relative to the probability distribution of the object's value, then the bidders start behaving passively. These results hold regardless of whether the floor is uncertain, the floor is certain or there is no floor.

In Theorem 5, we capture the impact on bids due to an imposition of a floor on the number of bidders. Precisely, we show that bidders' bids are lower under no floor situation than under uncertain and certain floor situations.

1.1 The literature

To the best of my knowledge, the present paper is the first to consider outside options which lead to uncertainty about the participation rate.

Kirchkamp et al. [6] consider outside options with a two-dimensional signal. However, they assume that the space of outside option is to the left of the value space on the real line, which leads to certainty about the number of participating bidders. Furthermore, there is no interdependence between the beliefs of the object's value and outside option. McAfee and McMillan [11] consider uncertain number of bidders. However, they do not model outside options and thus the signals are unidimensional.

Che and Gale [4] consider budget constraints in a two-dimensional model, where the first dimension draws the object's value and the second dimension draws the budget. In a recent paper, Dastidar and Agrawal [5] consider outside options as a divisible object where bidders have a fixed budget. If a bidder wins, he spends the remaining budget on outside option and induces utility from it. If he loses, he spends the entire budget on outside options.

Participation costs in auctions have been studied in Tan and Yilankaya [13], Cao and Tian [2], Cao et al. [3], etc. Tan and Yilankaya [13] characterize and show existence of an equilibrium of the second-price auction in cutoff strategies when the participation costs are common knowledge. Cao et al. [3] characterize and show existence of an equilibrium of the second-price auction in cutoff strategies when the participation costs are private information, i.e., bidders draw a two-dimensional signal: one for value and the other for cost. Cao and Tian [2] consider the first-price auction with common costs, characterize and prove existence of an equilibrium in cutoff strategies.

Our work is also related to the standard auctions literature without outside options. For example, Riley and Samuelson [12], Lebrun [9, 8], Maskin and Riley [10] and Landsberger et al. [7]. Riley and Samuelson [12] characterize a symmetric equilibrium while Lebrun [9] and Maskin and Riley [10] characterize an asymmetric equilibrium. Landsberger et al. [7] consider that the ordinal rank of the valuations is common knowledge between the bidders. Lebrun [8] captures the impact of a shift in the probability distribution of a bidder on bids.

The paper is organized as follows. Section 2 develops the model. Section 3 provides characterization properties of an equilibrium under uncertain floor, certain floor and no floor. Section 4 examines certain comparative properties. Section 5 concludes. The proofs are relegated to the appendix.

2 Economic model

Consider a first-price auction for the sale of an indivisible object with risk-neutral bidders. There is an outside option available with each bidder. The private information with the bidders is two-dimensional, i.e., bidders' draw the value of the object and its outside option from two different probability distributions. Every bidder participates for the auction if and only if their value of the object sold in the auction is greater than their outside option. However, the seller imposes a floor on the number of participants in order to conduct the auction, where the floor is uncertain. Therefore, there is uncertainty about the number of participants as well as about the floor imposed by the seller.

Let the set of potential bidders be $N = \{1, 2, \ldots, n+1\}$ where $n \in \mathcal{N}$. The value space is V_i and the space of outside option is T_i for every $i \in N$. Let $V_i = T_i = [0, a_i] \subset \Re_+$ for every $i \in N$. Given $v_i \in V_i$ and $t_i \in T_i$, the profile (v_i, t_i) is called bidder i's type.

The random variables of V_i and T_i are denoted by $\mathcal{V}_i$ and $\mathcal{T}_i$ respectively. The random variables are statistically independent with the probability distributions $F_i : V_i \to [0, 1]$ and $G_i : T_i \to [0, 1]$ along with the density functions f_i and g_i respectively for $\mathcal{V}_i$ and $\mathcal{T}_i$. Furthermore, the probability distributions are independent across bidders. We assume that the distribution functions are twice continuously differentiable and the density functions are atomless. We also assume that $F_i(x) = 0$ and $G_i(y) = 0$ for every $x \leq 0$ and $y \leq 0$.

Given $t_i \in T_i$, bidder i's conditional probability distribution of $\mathcal{V}_i$ is

$$F_i(v_i|\mathcal{V}_i \geq t_i) = \frac{F_i(v_i) - F_i(t_i)}{1 - F_i(t_i)}$$

for every $v_i \in [t_i, a_i]$ and for every $i \in N$. Given $v_i \in V_i$, bidder i's conditional probability distribution of $\mathcal{T}_i$ is

$$G_i(t_i|\mathcal{T}_i \leq v_i) = \frac{G_i(t_i)}{G_i(v_i)}$$

for every $t_i \in [0, v_i]$ and for every $i \in N$.

Let a random variable $\mathcal{Y}_i$ be defined as $\mathcal{Y}_i := \mathcal{V}_i - \mathcal{T}_i$ for every $i \in N$. Let the probability distribution of $\mathcal{Y}_i$ be denoted by $H_i : [-a_i, a_i] \to [0, 1]$ along with the density function h_i for every $i \in N$. The data on H_i is induced by F_i and G_i. Therefore, the conditional probability distribution of $\mathcal{Y}_i$ is

$$H_i(y_i|\mathcal{Y}_i \geq 0) = \frac{H_i(y_i) - H_i(0)}{1 - H_i(0)}$$

for every $y_i \in [0, a_i]$ and for every $i \in N$. Given $v_i, t_i \in [0, a_i]$, $H_i(v_i - t_i)$ is the probability that the difference between bidder i's value of the object and its outside option is lower than $v_i - t_i$ and $H_i(v_i - t_i|\mathcal{Y}_i \geq 0)$ is the probability that the difference between his object's value and its outside option is lower than $v_i - t_i$ conditional on i being a serious bidder.

Bidder $i \in N$ participates in the auction if and only if $v_i \geq t_i$. Otherwise, he does not participate. If a bidder participates, the bidder is called *active*. Otherwise, he is called *non-active*. Every bidder induces a belief about the number of active bidders from the probability distributions of values and outside options. Bidder $i \in N$ is active with probability $1 - H_i(0)$ and he is non-active with probability $H_i(0)$.

Let $\bar{N} := \{0, 1, 2, \dots, n\}$. Given $m \in \bar{N}$, let $N_i := \{j \in N : j \neq i\}$, $\mathbb{N}_i := \{A : A \subset N_i\}$ and $\mathbb{M}_i(m) := \{A \subset N_i : |A| = m\}$ for every $i \in N$. In words, N_i is the set of all the bidders but bidder i, $\mathbb{N}_i$ is the set of all the subsets of N_i and $\mathbb{M}_i(m)$ consists all the sets of N_i with cardinality of m. Note that $|\mathbb{N}_i| = 2^n$ and $|\mathbb{M}_i(m)| = \binom{n}{m}$ for every $i \in N$ and for every $m \in \bar{N}$.

Given $m \in \bar{N}$, let $q_i(m)$ be the belief of bidder $i \in N$ that m bidders are active. Since $\sum_{A \in \mathbb{N}_i} \prod_{j \in A}[1 - H_j(0)] \prod_{k \notin A} H_k(0) = 1$, we have

$$q_i(m) = \sum_{A \in \mathbb{M}_i(m)} \prod_{j \in A} [1 - H_j(0)] \prod_{k \notin A} H_k(0) \tag{1}$$

for every $m \in \bar{N}$ and $i \in N$.[1] Of course, $\sum_{m \in \bar{N}} q_i(m) = 1$ for every $i \in N$.

Given $A \in \mathbb{N}_i$, let $P_i(A)$ be the belief if bidder i that bidders in A are active, for every $i \in N$. Since $\sum_{A' \in \mathbb{N}_i} \prod_{j \in A'}[1 - H_j(0)] \prod_{k \notin A'} H_k(0) = 1$, we have

$$P_i(A) = \prod_{j \in A} [1 - H_j(0)] \prod_{k \notin A} H_k(0) \tag{2}$$

[1]Whenever we write $k \notin A$, it implies that $k \in N_i - A$.

Given $A \in \mathbb{N}_i$ and $m \in \bar{N}$ so that $|A| = m$, let $p_i(A, m)$ be the belief of bidder $i \in N$ that bidders in A are active conditional on m bidders being active. Then,

$$p_i(A, m) = \frac{\prod_{j \in A}[1 - H_j(0)] \prod_{k \notin A} H_k(0)}{\sum_{A' \in \mathbb{M}_i(m)} \prod_{j \in A'}[1 - H_j(0)] \prod_{k \notin A'} H_k(0)} \tag{3}$$

Clearly,

$$q_i(m) = \sum_{A \in \mathbb{M}_i(m)} P_i(A), \quad q_i(m)p_i(A, m) = P_i(A) \tag{4}$$

The seller imposes a floor on the number of participants and the auction is conducted if and only if the floor is met. The floor is private information to the seller and bidders can only observe a common prior on the floor which is denoted by $\bar{q} : N \to [0, 1]$. Given $m \in N$, $\bar{q}(m)$ is the probability that the seller sets a floor of m participants to conduct an auction. That is, $\bar{q}$ is a probability mass function of the floor. At the time when the floor is set, the seller does not know whether it will be met. It can be observed only when the bidders have submitted their bids. If the number of participants is lower than the floor, then the seller cancels the auction. For brevity, let

$$\bar{Q}(m) = \sum_{k=1}^{m} \bar{q}(k) \tag{5}$$

for every $m \in N$. Given $m \in N$, $\bar{Q}(m)$ is the probability that the floor is at most m. That is, $\bar{Q}$ is the cumulative distribution function of the floor. Then, $\bar{Q}(m + 1)q_i(m)$ is the belief of bidder $i \in N$ that $m + 1$ bidders are active (including bidder i) and the auction is conducted.

Lemma 1. *Given $i \in N$, the functions $H_i : [-a_i, a_i] \to [0, 1]$, $h_i : [-a_i, a_i] \to \Re_+$, $H_i(.|\mathcal{Y}_i \geq 0) : [0, a_i] \to [0, 1]$ and $h_i(.|\mathcal{Y}_i \geq 0) : [0, a_i] \to \Re_+$ are given by*

$$\begin{aligned}
H_i(y_i) &= \int_0^{a_i - y_i} F_i(t_i + y_i)g_i(t_i)dt_i + 1 - G_i(a_i - y_i) \\
h_i(y_i) &= \int_0^{a_i - y_i} f_i(t_i + y_i)g_i(t_i)dt_i \\
H_i(y_i|\mathcal{Y}_i \geq 0) &= \frac{1}{1 - \int_0^{a_i} g_i(t_i)F_i(t_i)dt_i}\bigg[\int_0^{a_i - y_i} [F_i(t_i + y_i) - F_i(t_i)] \\
&\qquad g_i(t_i)dt_i + \int_{a_i - y_i}^{a_i} [1 - F_i(t_i)]g_i(t_i)dt_i\bigg] \\
h_i(y_i|\mathcal{Y}_i \geq 0) &= \frac{\int_0^{a_i - y_i} f_i(t_i + y_i)g_i(t_i)dt_i}{1 - \int_0^{a_i} g_i(t_i)F_i(t_i)dt_i}
\end{aligned}$$

Let a Bayesian game be defined as

$$\Gamma = \langle N, \bar{q}, (V_i, T_i, q_i, p_i, F_i, G_i, H_i, u_i)_{i \in N} \rangle$$

where N is the set of potential bidders, $\bar{q}$ is the common prior about the floor on active bidders, V_i is the value space, T_i is the space of outside option, q_i is the belief about active bidders, p_i is the probability distribution of active bidders, F_i is the probability distribution of the object's value, G_i is the probability distribution of the outside option, H_i is the probability distribution of the difference between the object's value and outside option and is induced by F_i and G_i, $u_i(x) = x$ is the utility function, for every $i \in N$.

3 Equilibrium properties

Consider the first-price auction with uncertain floor. Let the bid function be denoted by $\sigma_i : V_i \times T_i \to \Re_+$. We assume that σ_i is strictly increasing in v_i, strictly decreasing in t_i and continuous. If a bidder draws a higher value for the object, he bids higher and if he draws a higher outside option, he bids lower.

Consider bidder $i \in N$ with type (v_i, t_i). Suppose he bids b while other bidders respect their bid functions. If $v_i < t_i$, then it is optimal for him not to participate. Suppose $v_i \geq t_i$. Let the unconditional bid distribution of bidder j be

$$\Omega_j(b) := \Pr[\sigma_j(\mathcal{V}_j, \mathcal{T}_j) \leq b]$$

and the conditional bid distribution be

$$\Omega_j(b|\mathcal{Y}_j \geq 0) := \Pr[\sigma_j(\mathcal{V}_j, \mathcal{T}_j) \leq b|\mathcal{Y}_j \geq 0]$$

Thus, the conditional expected utility function of bidder i is

$$\begin{aligned} U_i(v_i, t_i, b) = &\sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \prod_{j \in A} \Omega_j(b|\mathcal{Y}_j \geq 0) \\ &(v_i - b) + \left[1 - \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \prod_{j \in A} \Omega_j(b|\mathcal{Y}_j \geq 0)\right] t_i \end{aligned}$$

The first-order derivative is

$$\begin{aligned} D_b U_i(v_i, t_i, b) = &\sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) D_b \prod_{j \in A} \Omega_j(b|\mathcal{Y}_j \geq 0) \\ &(v_i - t_i - b) - \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \prod_{j \in A} \Omega_j(b|\mathcal{Y}_j \geq 0) \end{aligned}$$

In equilibrium, $\sigma_i(v_i, t_i) = b$. However, we cannot directly compute an inverse bid function because it is a two-dimensional case. Nonetheless, the following lemma allows us to do that.

Lemma 2. *Let $(\sigma_i)_{i \in N}$ be a Bayesian equilibrium when the floor is uncertain. Then, there exists a measurable function $\mu_i : [0, a_i] \to \Re_+$ so that*

$$\sigma_i(v_i, t_i) = \mu_i(v_i - t_i)$$

for every $v_i, t_i \in [0, a_i]$ with $v_i \geq t_i$ and for every $i \in N$.

Let the inverse of μ_i be denoted by π_i for every $i \in N$. Consequently,

$$\Omega_j(b|\mathcal{Y}_j \geq 0) = \Pr[\mu_j(\mathcal{Y}_j) \leq b|\mathcal{Y}_j \geq 0] = H_j[\pi_j(b)|\mathcal{Y}_j \geq 0]$$

For brevity, let

$$X(A, b) = \prod_{j \in A} H_j[\pi_j(b)|\mathcal{Y}_j \geq 0] \tag{6}$$

Then,

$$D_b X(A, b) = X(A, b) \sum_{j \in A} D_b \pi_j(b) \frac{h_j[\pi_j(b)|\mathcal{Y}_j \geq 0]}{H_j[\pi_j(b)|\mathcal{Y}_j \geq 0]} \tag{7}$$

In the following result, we characterize an equilibrium when the floor on the number of bidders is uncertain.

Theorem 1. *Let the floor be uncertain. A profile $(\sigma_i)_{i \in N}$ is a Bayesian equilibrium if and only if for every $i \in N$,*

$$\begin{aligned}
&[\pi_i(b) - b] \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) D_b X(A, b) \\
&= \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) X(A, b) \\
&\pi_i(0) = 0, \quad \pi_i(\bar{b}) = a_i \quad \textit{for some} \quad \bar{b} > 0
\end{aligned}$$

where $q_i(m)$, $p_i(A, m)$, $X(A, b)$ and $D_b X(A, b)$ are given in (1), (3), (6) and (7) respectively, for every $A \in \mathbb{M}_i(m)$, $m \in \bar{N}$ and $i \in N$. Furthermore, $H_j(.|\mathcal{Y}_j \geq 0)$ and $h_j(.|\mathcal{Y}_j \geq 0)$ are given in Lemma 1.

We say that bidders are symmetric if $F_i \triangleq F$, $G_i \triangleq G$, $a_i \triangleq a$, $q_i \triangleq q$ and $p_i \triangleq p$ for every $i \in N$.[2] Then,

[2]Whenever we talk about a symmetric equilibrium, we will suppress the subscripts.

$$q(m) = \binom{n}{m}[1 - H(0)]^m H(0)^{n-m}, \quad p(A, m) = \binom{n}{m}^{-1} \tag{8}$$

For brevity, let

$$Y(v - t, m) = H[v - t|\mathcal{Y} \geq 0]^m \tag{9}$$

Then,

$$D_{v-t}Y(v - t, m) = mY(v - t, m)\frac{h[v - t|\mathcal{Y} \geq 0]}{H[v - t|\mathcal{Y} \geq 0]} \tag{10}$$

In the next result, we characterize a symmetric equilibrium when the floor on the number bidders is uncertain.

Theorem 2. *Let the floor be uncertain. Let $F_i \triangleq F$, $G_i \triangleq G$, $a_i \triangleq a$, $q_i \triangleq q$ and $p_i \triangleq p$ for every $i \in N$. Then, σ is a symmetric Bayesian equilibrium if and only if*

$$D\mu(v - t) = \frac{\sum_{m \in \bar{N}} \bar{Q}(m + 1)q(m)D_{v-t}Y(v - t, m)}{\sum_{m \in \bar{N}} \bar{Q}(m + 1)q(m)Y(v - t, m)}[v - t - \mu(v - t)]$$

where $q(m)$, $Y(v - t, m)$ and $D_bY(v - t, m)$ are given in (8), (9) and (10) respectively. Furthermore, $H(.|\mathcal{Y} \geq 0)$ and $h(.|\mathcal{Y} \geq 0)$ are given in Lemma 1. Consequently,

$$\sigma(v, t) = v - t - \int_0^{v-t} \frac{\sum_{m \in \bar{N}} \bar{Q}(m + 1)q(m)Y(x, m)}{\sum_{m \in \bar{N}} \bar{Q}(m + 1)q(m)Y(v - t, m)}dx$$

for every $v \geq t$.

Now, consider that the floor is $m^* + 1$. Given $m^* \in \bar{N}$, let bidder i's bid function be denoted by $\sigma_i(., ., m^*) : V_i \times T_i \to \Re_+$. Analogous to Lemma 2, it can be shown that there exists a measurable function $\mu_i(., m^*) : [0, a_i] \to \Re_+$ so that

$$\sigma_i(v_i, t_i, m^*) = \mu_i(v_i - t_i, m^*) \tag{11}$$

for every $v_i, t_i \in [0, a_i]$ with $v_i \geq t_i$ and for every $i \in N$. Let the inverse of $\mu_i(., m^*)$ be denoted by $\pi_i(., m^*)$ for every $i \in N$. For brevity, let

$$Z(A, b, m^*) = \prod_{j \in A} H_j[\pi_j(b, m^*)|\mathcal{Y}_j \geq 0] \tag{12}$$

Then,

$$D_b Z(A,b,m^*) = Z(A,b,m^*) \sum_{j\in A} D_b\pi_j(b,m^*) \frac{h_j[\pi_j(b,m^*)|\mathcal{Y}_j \geq 0]}{H_j[\pi_j(b,m^*)|\mathcal{Y}_j \geq 0]} \quad (13)$$

The conditional expected utility function of bidder $i \in N$ is

$$U_i(v_i,t_i,b) = (v_i - t_i - b) \sum_{k=m^*}^{n} \sum_{A\in\mathbb{M}_i(k)} q_i(k)p_i(A,k)Z(A,b,m^*) + t_i$$

In the next result, we characterize an equilibrium when the floor on the number of bidders is certain.

Corollary 1. *Let the floor be m^*+1. A profile $(\sigma_i(.,.,m^*))_{i\in N}$ is a Bayesian equilibrium if and only if for every $i \in N$ and $m^* \in \bar{N}$,*

$$\pi_i(b,m^*) - b = \frac{\sum_{k=m^*}^{n} \sum_{A\in\mathbb{M}_i(k)} q_i(k)p_i(A,k)Z(A,b,m^*)}{\sum_{k=m^*}^{n} \sum_{A\in\mathbb{M}_i(k)} q_i(k)p_i(A,k)D_bZ(A,b,m^*)}$$
$$\pi_i(0,m^*) = 0, \quad \pi_i(\bar{b}(m^*),m^*) = a_i \quad \text{for some} \quad \bar{b}(m^*) > 0$$

where $p_i(A,m^*)$, $Z(A,b,m^*)$ and $D_bZ(A,b,m^*)$ are given in (3), (12) and (13) respectively. Furthermore, $H_j(.|\mathcal{Y}_j \geq 0)$ and $h_j(.|\mathcal{Y}_j \geq 0)$ are given in Lemma 1.

The proof of Corollary 1 is analogous to Theorem 1. Therefore, it is omitted.

The above characterization result can be simplified whenever $m = n$. This simplification is provided in the next result.

Corollary 2. *Let the floor be $n+1$. Then, a profile $(\sigma_i(.,.,n))_{i\in N}$ is a Bayesian equilibrium if and only if for every $i \in N$,*

$$D_b\pi_i(b,n) = \frac{H_i(\pi_i(b,n)|\mathcal{Y}_i \geq 0)}{h_i(\pi_i(b,n)|\mathcal{Y}_i \geq 0)} \left[\frac{1}{n}\sum_{k\in N} \frac{1}{\pi_k(b,n)-b} - \frac{1}{\pi_i(b,n)-b}\right]$$
$$\pi_i(0,n) = 0, \quad \pi_i(\bar{b}(n),n) = a_i \quad \text{for some} \quad \bar{b}(n) > 0$$

for every $i \in N$ where $H_i(.|\mathcal{Y}_i \geq 0)$ and $h_i(.|\mathcal{Y}_i \geq 0)$ are given in Lemma 1.

If $T_i = \{0\}$ for every $i \in N$, then the above corollary reduces to the standard asymmetric auctions without outside options that are given in Lebrun [9] and Maskin and Riley [10].

In the next result, we characterize a symmetric equilibrium when the floor on the number of bidders is certain.

Corollary 3. *Let the floor be $m^* + 1$. Let $F_i \triangleq F$, $G_i \triangleq G$, $a_i \triangleq a$ and $p_i \triangleq p$ for every $i \in N$. Then, $\sigma(.,.,m^*)$ is a symmetric Bayesian equilibrium if and only if for every $m^* \in \bar{N}$,*

$$D\mu(v-t, m^*) = \frac{\sum_{k=m^*}^{n} q(k) D_{v-t} Y(v-t, k)}{\sum_{k=m^*}^{n} q(k) Y(v-t, k)} [v - t - \mu(v-t, m^*)]$$

where $q(m)$, $Y(v-t, m)$ and $D_b Y(v-t, m)$ are given in (8), (9) and (10) respectively. Furthermore, $H(.|\mathcal{Y} \geq 0)$ and $h(.|\mathcal{Y} \geq 0)$ are given in Lemma 1. Consequently,

$$\sigma(v, t, m^*) = v - t - \int_0^{v-t} \frac{\sum_{k=m^*}^{n} q(k) Y(x, k)}{\sum_{k=m^*}^{n} q(k) Y(v-t, k)} dx$$

for every $v, t \in [0, a]$ so that $v \geq t$.

The proof of Corollary 3 is analogous to Theorem 2. Therefore, we omit it.

If $T_i = \{0\}$ for every $i \in N$, then the above characterization reduces to the standard symmetric auctions without outside options, that are given in Riley and Samuelson [12].

Now, consider that there is no floor. Let bidder i's bid function be denoted by $\sigma_i^* : V_i \times T_i \to \Re_+$. Analogous to Lemma 2, it can be shown that there exists a measurable function $\mu_i^* : [0, a_i] \to \Re_+$ so that

$$\sigma_i^*(v_i, t_i) = \mu_i^*(v_i - t_i) \tag{14}$$

for every $v_i, t_i \in [0, a_i]$ with $v_i \geq t_i$ and for every $i \in N$. Let the inverse of μ_i^* be denoted by π_i^* for every $i \in N$. Therefore, the conditional expected utility function is

$$\begin{aligned} U(v_i, t_i, b) &= (v_i - t_i - b) \sum_{A \in \mathbb{N}_i} P_i(A) \prod_{j \in A} H_j[\pi_j^*(b)|\mathcal{Y}_j \geq 0] + t_i \\ &= (v_i - t_i - b) \sum_{A \in \mathbb{N}_i} \prod_{j \in A} [H_j \circ \pi_j^*(b) - H_j(0)] \prod_{k \notin A} H_k(0) + t_i \\ &= (v_i - t_i - b) \prod_{j \in N_i} H_j \circ \pi_j^*(b) + t_i \end{aligned}$$

In the next result, we characterize an equilibrium when there is no floor on the number of bidders.

Corollary 4. *Let there be no floor. Then, $(\sigma^*)_{i \in N}$ is a Bayesian equilibrium if and only if*

$$D_b \pi_i^*(b) = \frac{H_i \circ \pi_i^*(b)}{h_i \circ \pi_i^*(b)} \left[\frac{1}{n} \sum_{k \in N} \frac{1}{\pi_k^*(b) - b} - \frac{1}{\pi_i^*(b) - b} \right]$$

$$\pi_i^*(0) = 0, \quad \pi_i^*(\bar{b}^*) = a_i \quad \textit{for some} \quad \bar{b}^* > 0$$

Under symmetry, the conditional expected utility function is

$$\begin{aligned} U(v,t,b) &= (v-t-b)\sum_{m\in\bar{N}} q(m)H[\pi^*(b)|\mathcal{Y}\geq 0]^m + t \\ &= (v-t-b)\sum_{m\in\bar{N}} \binom{n}{m}[H\circ\pi^*(b) - H(0)]^m H(0)^{n-m} + t \\ &= (v-t-b)H\circ\pi^*(b)^n + t \end{aligned}$$

In the next result, we characterize a symmetric equilibrium when there is no floor on the number of bidders.

Corollary 5. *Let there be no floor. Then, σ^* is a symmetric Bayesian equilibrium if and only if*

$$D\mu^*(v-t) = n\frac{h(v-t)}{H(v-t)}[v-t-\mu^*(v-t)]$$

where H and h are given in Lemma 1. Consequently,

$$\sigma^*(v,t) = v-t-\int_0^{v-t}\left[\frac{H(x)}{H(v-t)}\right]^n dx$$

for every $v, t \in [0,a]$ with $v \geq t$.

4 Comparative results

In this section, we examine certain comparative results.

Recall that the distribution function of value is F_i and that of outside option is G_i. Fixing G_i, suppose the distribution function of value shifts to $\hat{F}_i$ with the associated density $\hat{f}_i$. The new unconditional distribution function of $\mathcal{Y}_i$ is denoted by $\hat{H}_i$ along with the density $\hat{h}$ and the new conditional distribution of $\mathcal{Y}_i$ is denoted by $\hat{H}_i(.|\mathcal{Y}_i \geq 0)$ along with density $\hat{h}_i(.|\mathcal{Y}_i \geq 0)$.

Lemma 3. *Let $f_i/\hat{f}_i$ be non-decreasing and g_i be log-concave*[3] *for every $i \in N$. Then,*

1. *H_i reverse hazard rate dominates $\hat{H}_i$.*
2. *$H_i(.|\mathcal{Y}_i \geq 0)$ reverse hazard rate dominates $\hat{H}_i(.|\mathcal{Y}_i \geq 0)$.*

The above lemma states that if the distribution function of the object's value is strengthened with respect to monotone likelihood ratio dominance, then the conditional and unconditional distributions of the

[3] Bagnoli and Bergstrom [1] provide an elaborate set of properties on log-concavity of univariate probability distributions that are very useful in economic theory.

difference between object's value and its outside option are strengthened with respect to reverse hazard rate dominance.

Next, fixing F_i, suppose the probability distribution of outside option shifts to $\tilde{G}_i$ with associated density $\tilde{g}$. The new unconditional distribution function of $\mathcal{Y}_i$ is denoted by $\tilde{H}_i$ along with the density $\tilde{h}$ and the new conditional distribution of $\mathcal{Y}_i$ is denoted by $\tilde{H}_i(.|\mathcal{Y}_i \geq 0)$ along with density $\tilde{h}_i(.|\mathcal{Y}_i \geq 0)$.

Lemma 4. *Let $g_i/\tilde{g}_i$ be non-decreasing and f_i be log-concave for every $i \in N$. Then,*

1. *$\tilde{H}_i$ reverse hazard rate dominates H_i.*
2. *$\tilde{H}_i(.|\mathcal{Y}_i \geq 0)$ reverse hazard rate dominates $H_i(.|\mathcal{Y}_i \geq 0)$.*

The above two lemmas will be useful while analyzing comparative statics. In the next result, we capture the impact of perturbing the value distribution on the bids, when the floor is uncertain.

Theorem 3. *Consider two pairs of distribution functions: (F, G) and $(\hat{F}, G)$ so that $f/\hat{f}$ is non-decreasing and g is log-concave. Let σ be a symmetric Bayesian equilibrium under (F, G) and uncertain floor while $\hat{\sigma}$ be a Bayesian equilibrium under $(\hat{F}, G)$ and uncertain floor. Then, for every $v, t \in [0, a]$ so that $v \geq t$, we have*

$$\sigma(v, t) \geq \hat{\sigma}(v, t)$$

The next result captures the impact of perturbing the value distribution on the bids, when the floor is certain.

Corollary 6. *Consider two pairs of distribution functions: (F, G) and $(\hat{F}, G)$ so that $f/\hat{f}$ is non-decreasing and g is log-concave. Let $\sigma(., ., m^*)$ be a symmetric Bayesian equilibrium under (F, G) and a floor of $m^* + 1$ while $\hat{\sigma}(., ., m^*)$ be a Bayesian equilibrium under $(\hat{F}, G)$ and a floor of $m^* + 1$. Then, for every $v, t \in [0, a]$ so that $v \geq t$, we have*

$$\sigma(v, t, m^*) \geq \hat{\sigma}(v, t, m^*)$$

The proof of Corollary 6 is isomorphic to Theorem 3. Therefore, we omit it. The next result captures the impact of perturbing the value distribution on the bids, when there is no floor.

Corollary 7. *Consider two pairs of distribution functions: (F, G) and $(\hat{F}, G)$ so that $f/\hat{f}$ is non-decreasing and g is log-concave. Let σ^* be a symmetric Bayesian equilibrium under (F, G) no floor while $\hat{\sigma}^*$ be a Bayesian equilibrium under $(\hat{F}, G)$ and no floor. Then, for every $v, t \in [0, a]$ so that $v \geq t$, we have*

$$\sigma^*(v, t) \geq \hat{\sigma}^*(v, t)$$

The above properties say that if the probability distribution of object's value is strengthen relative to the outside option in terms of monotone likelihood ratio dominance, then bidders bid higher. As a result, the seller's revenue rises due to a rise in the degree of asymmetry between the object's value and its outside option.

In the next result, we capture the impact of perturbing the outside option distribution on the bids.

Theorem 4. *Consider two pairs of distribution functions: (F, G) and $(F, \tilde{G})$ so that $g/\tilde{g}$ is non-decreasing and f be log-concave. Let σ be a symmetric Bayesian equilibrium under (F, G) and uncertain floor while $\tilde{\sigma}$ be a Bayesian equilibrium under $(F, \tilde{G})$ and uncertain floor. Let $\sigma(.,.,m^*)$ be a symmetric Bayesian equilibrium under (F, G) and a floor of $m^* + 1$ while $\tilde{\sigma}(.,.,m^*)$ be a Bayesian equilibrium under $(F, \tilde{G})$ and a floor of $m^* + 1$. Let σ^* be a symmetric Bayesian equilibrium under (F, G) and no floor while $\tilde{\sigma}^*$ be a Bayesian equilibrium under $(F, \tilde{G})$ and no floor. Then, for every $v, t \in [0, a]$ so that $v \geq t$, we have*

1. $\sigma(v, t) \leq \tilde{\sigma}(v, t)$
2. $\sigma(v, t, m^*) \leq \tilde{\sigma}(v, t, m^*)$
3. $\sigma^*(v, t) \leq \tilde{\sigma}^*(v, t)$

The above property says that if the probability distribution of the outside option is strengthened relative to the object's value in terms of monotone likelihood ratio dominance, then bidders bid lower. In the next result, we compare the bid of no floor with the bids of uncertain and certain floor. The proof of Theorem 4 is isomorphic to Theorem 3. Therefore, it is omitted.

In the next result, we capture the impact of a floor on the number of bidders on bids.

Theorem 5. *Let σ be a symmetric Bayesian equilibrium under uncertain floor. Let $\sigma(.,.,m^*)$ be a symmetric Bayesian equilibrium under floor being $m^* + 1$. Let σ^* be a symmetric Bayesian equilibrium under no floor. Then, for every $v, t \in [0, a]$ with $v > t$, we have*

1. $\sigma^*(v, t) < \sigma(v, t)$
2. $\sigma^*(v, t) < \sigma(v, t, m^*)$

The above property says that bidders bid higher under uncertain floor and certain floor than under no floor, i.e., bidders' raise their bids due to the imposition of a floor on the number of bidders.

The next result provides a comparative result that concerns uncertain and certain floor on the number of bidders.

Theorem 6. *Let σ be a symmetric Bayesian equilibrium under uncertain floor. Given the floor to be $m^* + 1$, let $\sigma(.,.,m^*)$ be a symmetric Bayesian equilibrium.*

1. *If* $\bar{q}(m) = 0$ *for every* $m < m^* + 1$*, then* $\sigma(v, t) > \sigma(v, t, m^*)$ *for every* $v, t \in [0, a]$ *with* $v > t$*.*
2. *If* $\bar{q}(m) = 0$ *for every* $m > m^* + 1$*, then* $\sigma(v, t) < \sigma(v, t, m^*)$ *for every* $v, t \in [0, a]$ *with* $v > t$*.*
3. $\sigma(v, t, m^*+1) > \sigma(v, t, m^*)$*,* $U(v, t, \sigma(v, t, m^*)) > U(v, t, \sigma(v, t, m^*+1))$ *for every* $v, t \in [0, a]$ *with* $v > t$*.*

The above result is interpreted as follows. If the seller wishes to impose a floor of m^* and it is unlikely that bidders believe that the floor is lower than m^*, then bidders bid higher under uncertain floor than under certain floor. If the seller wishes to impose a floor of m^* and it is unlikely that bidders believe that the floor is higher than m^*, then bidders bid lower under uncertain floor than under certain floor. If the floor on the number of bidders rises, bidders bid higher. However, their expected utilities decline.

5 Conclusion

In this paper, we have considered the sale of an indivisible object when every potential bidder has an outside option. Since the participation of the bidders depend on whether their value for the object exceeds their outside option, it leads to uncertain number of participants. A floor on the number of participants is imposed by the seller in order to conduct the auction. Thus, there is uncertainty about the number of participants and whether the auction is conduced.

We have characterized an equilibrium, when the floor is uncertain, the floor is certain and there is no floor. We have shown that the bidders bid higher due to the strengthening of value distribution or weakening of outside option distribution. We have also shown that bids are lower in the absence of floor than when the floor exists.

Appendix A: Proofs

Proof of Lemma 1. By definition,

$$H_i(0) = \Pr(\mathcal{Y}_i \leq 0) = \int_0^{a_i} \int_0^{t_i} f_i(v_i) g_i(t_i) dv_i dt_i = \int_0^{a_i} g_i(t_i) F_i(t_i) dt_i$$

and

$$H_i(y_i) = \Pr(\mathcal{Y}_i \leq y_i) = \Pr(\mathcal{V}_i \leq \mathcal{T}_i + y_i) = \int_0^{a_i} \Pr(\mathcal{V}_i \leq t_i + y_i) g_i(t_i) dt_i$$

If $t_i + y_i \leq a_i$, then $\Pr(\mathcal{V}_i \leq t_i + y_i) = F_i(t_i + y_i)$. If $t_i + y_i > a_i$, then $\Pr(\mathcal{V}_i \leq t_i + y_i) = 1$. So, we have

$$\begin{aligned} H_i(y_i) &= \int_0^{a_i - y_i} F_i(t_i + y_i) g_i(t_i) dt_i + \int_{a_i - y_i}^{a_i} g_i(t_i) dt_i \\ &= \int_0^{a_i - y_i} F_i(t_i + y_i) g_i(t_i) dt_i + 1 - G_i(a_i - y_i) \end{aligned}$$

This implies

$$\begin{aligned} H_i(y_i) - H_i(0) &= \int_0^{a_i - y_i} F_i(t_i + y_i) g_i(t_i) dt_i + 1 - G_i(a_i - y_i) \\ &\quad - \int_0^{a_i} g_i(t_i) F_i(t_i) dt_i \\ &= \int_0^{a_i - y_i} [F_i(t_i + y_i) - F_i(t_i)] g_i(t_i) dt_i \\ &\quad - \int_{a_i - y_i}^{a_i} F_i(t_i) g_i(t_i) dt_i + \int_{a_i - y_i}^{a_i} g_i(t_i) dt_i \\ &= \int_0^{a_i - y_i} [F_i(t_i + y_i) - F_i(t_i)] g_i(t_i) dt_i \\ &\quad + \int_{a_i - y_i}^{a_i} [1 - F_i(t_i)] g_i(t_i) dt_i \end{aligned}$$

Thus, the result follows. ∎

Proof of Lemma 2. Consider bidder $i \in N$ with type (v_i, t_i). The conditional expected utility function can be rewritten as

$$\begin{aligned} U_i(v_i, t_i, b) = \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} & \bar{Q}(m+1) q_i(m) p_i(A, m) \prod_{j \in A} \Omega_j(b | \mathcal{Y}_j \geq 0) \\ & (v_i - t_i - b) + t_i \end{aligned}$$

Let $b^* \in \arg\max_b U_i(v_i, t_i, b)$. Then, $b^* \in \arg\max_b U_i(v_i, t_i, b) - t_i$. Consider $\max_b U_i(v_i, t_i, b) - t_i$ which equals

$$\max_b \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \prod_{j \in A} \Omega_j(b | \mathcal{Y}_j \geq 0)(z_i - b)$$

where $z_i = v_i - t_i$. Since the two optimization problems are equivalent and the bid functions are strictly increasing and continuous, there exists a unique optimal bid for every z_i. This implies that there exists a function $\mu_i : [0, a_i] \to \Re_+$ so that $b = \mu_i(z_i)$ in an equilibrium. Thus, the result holds. ∎

Proof of Theorem 1. Consider bidder $i \in N$ with type (v_i, t_i) and bid b. From Lemma 2, the conditional expected utility function can be rewritten as

$$U_i(v_i, t_i, b) = \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) X(A, b)(v_i - t_i - b) + t_i$$

Then,

$$\begin{aligned} D_b U_i(v_i, t_i, b) = \Bigg[&(v_i - t_i - b) \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \\ &D_b X(A, b) - \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) X(A, b) \Bigg] \end{aligned}$$

In equilibrium, $v_i - t_i = \pi_i(b)$ and thus we have

$$\begin{aligned} &[\pi_i(b) - b] \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) D_b X(A, b) \\ &= \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) X(A, b) \end{aligned}$$

Conversely, suppose $(\pi_i)_{i \in N}$ solves the differential equations given in the present proposition. Consider bidder $i \in N$ with type (v, t) with $v > t$ and bid b. Suppose he overbids to c so that $\pi_i(c) > v - t$. Then, from the first-order derivative, we have

$$\begin{aligned} D_c U_i(v_i, t_i, c) = \Bigg[&(v_i - t_i - c) \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \\ &D_b X(A, c) - \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) X(A, c) \Bigg] \\ D_c U_i(v_i, t_i, c) < \Bigg[&(\pi_i(c) - c) \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) \\ &D_b X(A, c) - \sum_{m \in \bar{N}} \sum_{A \in \mathbb{M}_i(m)} \bar{Q}(m+1) q_i(m) p_i(A, m) X(A, c) \Bigg] \\ &= 0 \end{aligned}$$

Thus, overbids are not profitable. Similarly, we can show that underbids are also not profitable. Hence, $(\sigma_i)_{i \in N}$ is a Bayesian equilibrium. ■

Proof of Theorem 2. Consider a bidder with type (v,t) and bid b. The conditional expected utility function is

$$U(v,t,b) = \sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(\pi(b),m)(v-t-b)+t$$

where $Y(\pi(b),m) = H[\pi(b)|\mathcal{Y}\geq 0]^m$. Then,

$$\begin{aligned}D_bU(v,t,b) = (v-t-b)&\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_bY(\pi(b),m)\\ &-\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(\pi(b),m)\end{aligned}$$

where

$$D_bY(\pi(b),m) = mY(\pi(b),m)D_b\pi(b)\frac{h[\pi(b)|\mathcal{Y}\geq 0]}{H[\pi(b)|\mathcal{Y}\geq 0]}$$

In equilibrium, $v-t=\pi(b)$ and thus we have

$$\pi(b)-b = \frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(\pi(b),m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_bY(\pi(b),m)}$$

From (9) and (10), the first-order differential equation can be rewritten as

$$\begin{aligned}&D_b\pi(b)[\pi(b)-b]\sum_{m\in\bar{N}} m\bar{Q}(m+1)q(m)Y(\pi(b),m)\frac{h[\pi(b)|\mathcal{Y}\geq 0]}{H[\pi(b)|\mathcal{Y}\geq 0]}\\ &=\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(\pi(b),m)\end{aligned}$$

Using $b=\mu(v-t)$, we have

$$\begin{aligned}&[v-t-\mu(v-t)]\sum_{m\in\bar{N}} m\bar{Q}(m+1)q(m)Y(v-t,m)\frac{h[v-t|\mathcal{Y}\geq 0]}{H[v-t|\mathcal{Y}\geq 0]}\\ &=D_{v-t}\mu(v-t)\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)\end{aligned}$$

which is equivalent to

$$\begin{aligned}&[v-t-\mu(v-t)]\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_{v-t}Y(v-t,m)\\ &=D_{v-t}\mu(v-t)\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)\end{aligned}$$

Using the fundamental theorem of calculus with $\mu(0) = 0$, we have

$$\sigma(v,t) = \frac{\int_0^{v-t} \sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)xD_xY(x,m)dx}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)}$$

Using integration-by-parts, we have

$$\sigma(v,t) = v - t - \int_0^{v-t} \frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(x,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)}dx$$

The proof of the converse is similar to Theorem 1. ■

Proof of Corollary 2. Consider bidder $i \in N$ with type (v_i, t_i) and bid b. If $m = n$, $\mathbb{M}_i(m) = \{N_i\}$ and $p(N_i, n) = 1$. Consequently,

$$Z(A,b,n) = \prod_{j\in N_i} H_j[\pi_j(b,n)|\mathcal{Y}_j \geq 0]$$

and

$$\frac{D_bZ(A,b,n)}{Z(A,b,n)} = \sum_{j\in N_i} D_b\pi_j(b,n)\frac{h_j[\pi_j(b,n)|\mathcal{Y}_j \geq 0]}{H_j[\pi_j(b,n)|\mathcal{Y}_j \geq 0]}$$

for every $b > 0$ and $A \in \mathbb{M}_i(n)$. The first-order condition reduces to

$$[\pi_i(b,n) - b]D_bZ(A,b,n) = Z(A,b,n)$$

which can be rewritten as

$$\frac{1}{\pi_i(b,n) - b} = \sum_{j\in N_i} D_b\pi_j(b,n)\frac{h_j(\pi_j(b,n)|\mathcal{Y}_j \geq 0)}{H_j(\pi_j(b,n)|\mathcal{Y}_j \geq 0)} \tag{15}$$

Summing for every $i \in N$, we have

$$\begin{aligned}\sum_{k\in N} \frac{1}{\pi_k(b,n) - b} &= n\sum_{j\in N} D_b\pi_j(b,n)\frac{h_j(\pi_j(b,n)|\mathcal{Y}_j \geq 0)}{H_j(\pi_j(b,n)|\mathcal{Y}_j \geq 0)} \\ &= n\Bigg[D_b\pi_i(b,n)\frac{h_i(\pi_i(b)|\mathcal{Y}_i \geq 0)}{H_i(\pi_i(b,n)|\mathcal{Y}_i \geq 0)} + \\ &\quad \sum_{j\in N_i} D_b\pi_j(b,n)\frac{h_j(\pi_j(b,n)|\mathcal{Y}_j \geq 0)}{H_j(\pi_j(b,n)|\mathcal{Y}_j \geq 0)}\Bigg]\end{aligned}$$

From (15), we have

$$\sum_{k\in N}\frac{1}{\pi_k(b,n)-b} = n\left[D_b\pi_i(b,n)\frac{h_i(\pi_i(b,n)|\mathcal{Y}_i\geq 0)}{H_i(\pi_i(b,n)|\mathcal{Y}_i\geq 0)} + \frac{1}{\pi_i(b,n)-b}\right]$$

which is equivalent to

$$D_b\pi_i(b,n) = \frac{H_i(\pi_i(b,n)|\mathcal{Y}_i\geq 0)}{h_i(\pi_i(b,n)|\mathcal{Y}_i\geq 0)}\left[\frac{1}{n}\sum_{k\in N}\frac{1}{\pi_k(b,n)-b} - \frac{1}{\pi_i(b,n)-b}\right]$$

The converse follows directly from Corollary 1. ■

Proof of Lemma 3. Consider bidder $i\in N$. Given $y_i\in[-a_i,a_i]$, let

$$J_i(y_i) = \frac{H_i(y_i)}{\hat{H}_i(y_i)}$$

Then,

$$DJ_i(y_i) = \frac{h_i(y_i)\hat{H}_i(y_i) - \hat{h}_i(y_i)H_i(y_i)}{[\hat{H}_i(y_i)]^2}$$

Since

$$H_i(y_i|\mathcal{Y}_i\geq 0) = \frac{H_i(y_i)-H_i(0)}{1-H_i(0)} \text{ and } \hat{H}_i(y_i|\mathcal{Y}_i\geq 0) = \frac{\hat{H}_i(y_i)-\hat{H}_i(0)}{1-\hat{H}_i(0)},$$

we have

$$\frac{H_i(y_i|\mathcal{Y}_i\geq 0)}{\hat{H}_i(y_i|\mathcal{Y}_i\geq 0)} = \frac{H_i(y_i)-H_i(0)}{\hat{H}_i(y_i)-\hat{H}_i(0)}\frac{1-\hat{H}_i(0)}{1-H_i(0)} \triangleq K_i(y_i)$$

Then,

$$DK_i(y_i) = \frac{[\hat{H}_i(y_i)-\hat{H}_i(0)]h_i(y_i) - [H_i(y_i)-H_i(0)]\hat{h}_i(y_i)}{[\hat{H}_i(y_i)-\hat{H}_i(0)]^2}\frac{1-\hat{H}_i(0)}{1-H_i(0)}$$

To show that H_i reverse hazard rate dominates $\hat{H}_i$, we need to show that

$$\hat{H}_i(y_i)h_i(y_i) - \hat{h}_i(y_i)H_i(y_i) \geq 0$$

To show that $H(.|\mathcal{Y}_i\geq 0)$ reverse hazard rate dominates $\hat{H}(.|\mathcal{Y}_i\geq 0)$, we need to show that

$$[\hat{H}_i(y_i)-\hat{H}_i(0)]h_i(y_i) \geq [H_i(y_i)-H_i(0)]\hat{h}_i(y_i)$$

It is useful to show that $h_i/\hat{h}_i$ is non-decreasing. Note that $h_i(y_i) = \int_0^{a_i-y_i} f_i(y_i+t_i)g_i(t_i)dt_i$ and $\hat{h}_i(y_i) = \int_0^{a_i-y_i}\hat{f}_i(y_i+t_i)g_i(t_i)dt_i$. Let $z_i =$

$y_i + t_i$, we have $h_i(y_i) = \int_{y_i}^{a_i} f_i(z_i) g_i(z_i - y_i) dz_i$. Since $g_i(z_i - y_i) = 0$ for every $z_i \leq y_i$, we can rewrite h_i as

$$\begin{aligned} h_i(y_i) &= \int_{y_i}^{a_i} f_i(z_i) g_i(z_i - y_i) dz_i + \int_0^{y_i} f_i(z_i) g_i(z_i - y_i) dz_i \\ &= \int_0^{a_i} f_i(z_i) g_i(z_i - y_i) dz_i \end{aligned}$$

Similarly, $\hat{h}_i(y_i) = \int_0^{a_i} \hat{f}_i(z_i) g_i(z_i - y_i) dz_i$. Consider $y_i, y_i' \in [-a_i, a_i]$ so that $y_i \leq y_i'$. We show that

$$\frac{h_i(y_i)}{\hat{h}_i(y_i)} \leq \frac{h_i(y_i')}{\hat{h}_i(y_i')}$$

which is equivalent to show that $h_i(y_i') \hat{h}_i(y_i) - h_i(y_i) \hat{h}_i(y_i') \geq 0$. To save on notations, let $\varphi_i(y_i, y_i') := h_i(y_i') \hat{h}_i(y_i) - h_i(y_i) \hat{h}_i(y_i')$. We show that $\varphi_i(y_i, y_i') \geq 0$. Using the expressions of h_i and $\hat{h}_i$, we have

$$\begin{aligned} \varphi_i(y_i, y_i') &= \left[\int_0^{a_i} f_i(z_i') g_i(z_i' - y_i') dz_i' \right] \left[\int_0^{a_i} \hat{f}_i(z_i) g_i(z_i - y_i) dz_i \right] \\ &\quad - \left[\int_0^{a_i} f_i(z_i) g_i(z_i - y_i) dz_i \right] \left[\int_0^{a_i} \hat{f}_i(z_i') g_i(z_i - y_i') dz_i' \right] \\ &= \int_0^{a_i} \int_0^{a_i} f_i(z_i') \hat{f}_i(z_i) g_i(z_i' - y_i') g_i(z_i - y_i) dz_i dz_i' \\ &\quad - \int_0^{a_i} \int_0^{a_i} f_i(z_i) \hat{f}_i(z_i') g_i(z_i' - y_i') g_i(z_i - y_i) dz_i' dz_i \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \varphi_i(y_i, y_i') &= \int_0^{a_i} \int_0^{a_i} f_i(z_i') \hat{f}_i(z_i) g_i(z_i' - y_i') g_i(z_i - y_i) dz_i dz_i' \\ &\quad - \int_0^{a_i} \int_0^{a_i} f_i(z_i') \hat{f}_i(z_i) g_i(z_i - y_i') g_i(z_i' - y_i) dz_i dz_i' \\ &= \int_0^{a_i} \int_0^{a_i} f_i(z_i') \hat{f}_i(z_i) [g_i(z_i' - y_i') g_i(z_i - y_i) \\ &\quad - g_i(z_i - y_i') g_i(z_i' - y_i)] dz_i dz_i' \end{aligned} \tag{16}$$

By changing the roles of z_i and z_i', we have

$$\begin{aligned} \varphi_i(y_i, y_i') &= \int_0^{a_i} \int_0^{a_i} f_i(z_i) \hat{f}_i(z_i') [g_i(z_i - y_i') g_i(z_i' - y_i) \\ &\quad - g_i(z_i' - y_i') g_i(z_i - y_i)] dz_i dz_i' \end{aligned} \tag{17}$$

Summing (16) and (17), we get

$$\begin{aligned}2\varphi_i(y_i, y_i') &= \int_0^{a_i}\int_0^{a_i} [f_i(z_i)\hat{f}_i(z_i') - f_i(z_i')\hat{f}_i(z_i)] \\ &\quad [g_i(z_i - y_i')g_i(z_i' - y_i) - g_i(z_i' - y_i')g_i(z_i - y_i)]dz_i dz_i' \\ &= \int_0^{a_i}\int_0^{z_i'} [f_i(z_i)\hat{f}_i(z_i') - f_i(z_i')\hat{f}_i(z_i)] \\ &\quad [g_i(z_i - y_i')g_i(z_i' - y_i) - g_i(z_i' - y_i')g_i(z_i - y_i)]dz_i dz_i' \\ &+ \int_0^{a_i}\int_{z_i'}^{a_i} [f_i(z_i)\hat{f}_i(z_i') - f_i(z_i')\hat{f}_i(z_i)] \\ &\quad [g_i(z_i - y_i')g_i(z_i' - y_i) - g_i(z_i' - y_i')g_i(z_i - y_i)]dz_i dz_i'\end{aligned}$$

Using Fubini's theorem, we have

$$\begin{aligned}2\varphi_i(y_i, y_i') &= \int_0^{a_i}\int_0^{z_i'} [f_i(z_i)\hat{f}_i(z_i') - f_i(z_i')\hat{f}_i(z_i)] \\ &\quad [g_i(z_i - y_i')g_i(z_i' - y_i) - g_i(z_i' - y_i')g_i(z_i - y_i)]dz_i dz_i' \\ &+ \int_0^{a_i}\int_0^{z_i'} [f_i(z_i')\hat{f}_i(z_i) - f_i(z_i)\hat{f}_i(z_i')] \\ &\quad [g_i(z_i' - y_i')g_i(z_i - y_i) - g_i(z_i - y_i')g_i(z_i' - y_i)]dz_i dz_i'\end{aligned}$$

which implies

$$\begin{aligned}\varphi_i(y_i, y_i') &= \int_0^{a_i}\int_0^{z_i'} [f_i(z_i)\hat{f}_i(z_i') - f_i(z_i')\hat{f}_i(z_i)] \\ &\quad [g_i(z_i - y_i')g_i(z_i' - y_i) - g_i(z_i' - y_i')g_i(z_i - y_i)]dz_i dz_i'\end{aligned}$$

Since $z_i \leq z_i'$ and $f_i/\hat{f}_i$ is non-decreasing, we have

$$\frac{f_i(z_i)}{\hat{f}_i(z_i)} \leq \frac{f_i(z_i')}{\hat{f}_i(z_i')}$$

which implies $f_i(z_i)\hat{f}_i(z_i') - f_i(z_i')\hat{f}_i(z_i) \leq 0$. To show $\varphi_i(y_i, y_i') \geq 0$, it suffices to show that $g_i(z_i - y_i')g_i(z_i' - y_i) \leq g_i(z_i' - y_i')g_i(z_i - y_i)$.

Let $x_i = z_i - y_i$, $x_i' = z_i' - y_i'$, $w_i = z_i - y_i'$ and $w_i' = z_i' - y_i$. Then, $x_i + x_i' = w_i + w_i'$. We show that $g_i(w_i)g_i(w_i') \leq g_i(x_i)g_i(x_i')$. Since $y_i \leq y_i'$ and $z_i \leq z_i'$, we have $w_i \leq x_i \leq w_i'$ and $w_i \leq x_i' \leq w_i'$. Then, there exists $\lambda_1, \lambda_2 \in [0, 1]$ so that $x_i = \lambda_1 w_i + (1-\lambda_1)w_i'$ and $x_i' = \lambda_2 w_i + (1-\lambda_2)w_i'$.

Since g is log-concave, we have

$$\ln g(x_i) \geq \lambda_1 \ln g(w_i) + (1-\lambda_1)\ln g(w_i')$$

and

$$\ln g(x_i') \geq \lambda_2 \ln g(w_i) + (1-\lambda_2) \ln g(w_i')$$

Summing the above two inequalities and using $\lambda_1 + \lambda_2 = 1$, we get

$$\begin{aligned} \ln g(x_i) + \ln g(x_i') &\geq \lambda_1 \ln g(w_i) + (1-\lambda_1) \ln g(w_i') \\ &\quad + \lambda_2 \ln g(w_i) + (1-\lambda_2) \ln g(w_i') \\ &= \ln g(w_i) + \ln g(w_i') \end{aligned}$$

which implies $g(x_i)g(x_i') \geq g(w_i)g(w_i')$. Therefore,

$$\frac{h_i(y_i)}{\hat{h}_i(y_i)} \leq \frac{h_i(y_i')}{\hat{h}_i(y_i')}$$

which implies that $h_i/\hat{h}_i$ is non-decreasing.

We show H_i reverse hazard rate dominates $\hat{H}_i$. To do so, it suffices to show that $H_i/\hat{H}_i$ is non-decreasing which is equivalent to $\hat{H}_i(y_i)h_i(y_i) - \hat{h}_i(y_i)H_i(y_i) \geq 0$. This is equivalent to

$$h_i(y_i) \int_{-a_i}^{y_i} \hat{h}_i(r_i)dr_i - \hat{h}_i(y_i) \int_{-a_i}^{y_i} h_i(r_i)dr_i \geq 0$$

which implies

$$\int_{-a_i}^{y_i} [h_i(y_i)\hat{h}_i(r_i) - \hat{h}_i(y_i)h_i(r_i)]dr_i \geq 0$$

The above inequality is true as $r_i \leq y_i$ and $h_i/\hat{h}_i$ is non-decreasing. Thus, H_i reverse hazard rate dominates $\hat{H}_i$.

We now show that $H_i(.|\mathcal{Y}_i \geq 0)$ reverse hazard rate rate dominates $\hat{H}_i(.|\mathcal{Y}_i \geq 0)$, i.e., we show that

$$[\hat{H}_i(y_i) - \hat{H}_i(0)]h_i(y_i) - [H_i(y_i) - H_i(0)]\hat{h}_i(y_i) \geq 0$$

which is equivalent to show that

$$h_i(y_i) \int_{0}^{y_i} \hat{h}_i(r_i)dr_i - \hat{h}_i(y_i) \int_{0}^{y_i} h_i(r_i)dr_i \geq 0$$

which is equivalent to show that

$$\int_{0}^{y_i} [h_i(y_i)\hat{h}_i(r_i) - \hat{h}_i(y_i)h_i(r_i)]dr_i \geq 0$$

The above inequality is true as $r_i \leq y_i$ and $h_i/\hat{h}_i$ is non-decreasing. Thus, $H(.|\mathcal{Y}_i \geq 0)$ reverse hazard rate dominates $\hat{H}(.|\mathcal{Y}_i \geq 0)$. ■

Proof of Lemma 4. We can rewrite $H_i(0)$ as

$$\begin{aligned}H_i(0) &= \int_0^{a_i}\int_{v_i}^{a_i} f_i(v_i)g_i(t_i)dt_idv_i \\ &= \int_0^{a_i} f_i(v_i)[1-G_i(v_i)]dv_i\end{aligned}$$

and $H_i(y_i)$ as

$$\begin{aligned}H_i(y_i) &= \Pr(\mathcal{T}_i \geq \mathcal{V}_i - y_i) \\ &= \int_0^{a_i} \Pr(\mathcal{T}_i \geq v_i - y_i)f_i(v_i)dv_i\end{aligned}$$

If $v_i - y_i \leq 0$, $\Pr(\mathcal{T}_i \geq \mathcal{V}_i - y_i) = 1$. If $0 < v_i - y_i < a_i$, $\Pr(\mathcal{T}_i \geq \mathcal{V}_i - y_i) = 1 - G_i(v_i - y_i)$. If $v_i - y_i \geq a_i$, $\Pr(\mathcal{T}_i \geq \mathcal{V}_i - y_i) = 0$. This implies

$$H_i(y_i) = \int_0^{a_i}[1 - G_i(v_i - y_i)]f_i(v_i)dv_i$$

Consequently,

$$H_i(y_i|\mathcal{Y}_i \geq 0) = \frac{\int_0^{a_i}[G_i(v_i) - G_i(v_i - y_i)]f_i(v_i)dv_i}{1 - \int_0^{a_i}[1 - G_i(v_i)]f_i(v_i)dv_i}$$

and

$$h_i(y_i|\mathcal{Y}_i \geq 0) = \frac{\int_0^{a_i} g_i(v_i - y_i)f_i(v_i)dv_i}{1 - \int_0^{a_i}[1 - G_i(v_i)]f_i(v_i)dv_i}$$

From the expression of $H_i(y_i)$, we have $h_i(y_i) = \int_0^{a_i} g_i(v_i - y_i)f_i(v_i)dv_i$. Let $u_i = v_i - y_i$. Then,

$$\begin{aligned}h_i(y_i) &= \int_{-y_i}^{a_i - y_i} g_i(u_i)f_i(u_i + y_i)du_i \\ &= \int_0^{a_i} g_i(u_i)f_i(u_i + y_i)du_i\end{aligned}$$

as $f_i(u_i + y_i) = 0$ for every $y_i + u_i \geq a_i$ and $g_i(u_i) = 0$ for every $u_i \leq 0$. Similarly, $\tilde{h}(y_i) = \int_0^{a_i} \tilde{g}_i(u_i)f_i(u_i + y_i)du_i$.

Consider $y_i, y_i' \in [-a_i, a_i]$ so that $y_i \leq y_i'$ and let

$$\phi_i(y_i, y_i') = h_i(y_i')\tilde{h}_i(y_i) - h_i(y_i)\tilde{h}_i(y_i')$$

We show that $\phi_i(y_i, y_i') \leq 0$. Using the expressions of h_i and $\tilde{h}_i$, we have

$$\phi_i(y_i, y_i') = \int_0^{a_i} \int_0^{a_i} g_i(u_i)\tilde{g}_i(u_i')[f_i(u_i + y_i')f_i(u_i' + y_i) \\ - f_i(u_i + y_i)f_i(u_i' + y_i')]du_i du_i'$$

By changing the roles of u_i and u_i', we can rewrite the above equation as

$$\phi_i(y_i, y_i') = \int_0^{a_i} \int_0^{a_i} g_i(u_i')\tilde{g}_i(u_i)[f_i(u_i' + y_i')f_i(u_i + y_i) \\ - f_i(u_i' + y_i)f_i(u_i + y_i')]du_i du_i'$$

Summing the above two equations, we have

$$2\phi_i(y_i, y_i') = \int_0^{a_i} \int_0^{a_i} [g_i(u_i')\tilde{g}_i(u_i) - g_i(u_i)\tilde{g}_i(u_i')][f_i(u_i + y_i)f_i(u_i' + y_i') \\ - f_i(u_i + y_i')f_i(u_i' + y_i)]du_i du_i'$$

which implies

$$\phi_i(y_i, y_i') = \int_0^{a_i} \int_{u_i'}^{a_i} [g_i(u_i')\tilde{g}_i(u_i) - g_i(u_i)\tilde{g}_i(u_i')][f_i(u_i + y_i)f_i(u_i' + y_i') \\ - f_i(u_i + y_i')f_i(u_i' + y_i)]du_i du_i'$$

Since $u_i \geq u_i'$, we have

$$\frac{g_i(u_i)}{\tilde{g}_i(u_i)} \geq \frac{g_i(u_i')}{\tilde{g}_i(u_i')}$$

which implies $g_i(u_i')\tilde{g}_i(u_i) - g_i(u_i)\tilde{g}_i(u_i') \leq 0$. It remains to show that $f_i(u_i + y_i)f_i(u_i' + y_i') - f_i(u_i + y_i')f_i(u_i' + y_i) \geq 0$.

Let $x_i = u_i + y_i$, $x_i' = u_i' + y_i'$, $w_i = u_i + y_i'$ and $w_i' = u_i' + y_i$. Then, $x_i + x_i' = w_i + w_i'$, $w_i' \leq x_i \leq w_i$ and $w_i' \leq x_i' \leq w_i$. This implies that there exists $\lambda_1, \lambda_2 \in [0, 1]$ so that $x_i = \lambda_1 w_i' + (1 - \lambda_1)w_i$ and $x_i' = \lambda_2 w_i' + (1 - \lambda_2)w_i$.

Since f is log-concave, we have

$$\ln f_i(x_i) \geq \lambda_1 \ln f_i(w_i') + (1 - \lambda_1) \ln f_i(w_i)$$

and

$$\ln f_i(x_i') \geq \lambda_2 \ln f_i(w_i') + (1 - \lambda_2) \ln f_i(w_i)$$

Since $\lambda_1 + \lambda_2 = 1$, adding the above two inequalities gives $f_i(x_i)f_i(x_i') - f_i(w_i)f_i(w_i') \geq 0$. Therefore, $\phi_i(y_i, y_i') \leq 0$. Thus, $h_i/\tilde{h}_i$ is non-increasing.

We show $\tilde{H}_i$ reverse hazard rate dominates H_i. To do so, it suffices to show that $H_i/\tilde{H}_i$ is non-increasing which is equivalent to $\tilde{H}_i(y_i)h_i(y_i) - \tilde{h}_i(y_i)H_i(y_i) \leq 0$. This is equivalent to

$$\int_{-a_i}^{y_i} [h_i(y_i)\tilde{h}_i(r_i) - \tilde{h}_i(y_i)h_i(r_i)]dr_i \leq 0$$

The above inequality is true as $r_i \leq y_i$ and $h_i/\tilde{h}_i$ is non-increasing. Thus, $\tilde{H}_i$ reverse hazard rate dominates H_i.

Finally, to show that $\tilde{H}_i(.|\mathcal{Y}_i \geq 0)$ reverse hazard rate dominates $H_i(.|\mathcal{Y}_i \geq 0)$, it suffices to show that

$$[\tilde{H}_i(y_i) - \tilde{H}_i(0)]h_i(y_i) \leq [H_i(y_i) - H_i(0)]\tilde{h}_i(y_i)$$

This is equivalent to

$$\int_{0}^{y_i} [h_i(y_i)\tilde{h}_i(r_i) - \tilde{h}_i(y_i)h_i(r_i)]dr_i \leq 0$$

which is true as $r_i \leq y_i$ and $h_i/\tilde{h}_i$ is non-increasing. This completes the proof. ■

Proof of Theorem 3. Consider that the floor is uncertain. From Lemma 3, H reverse hazard rate dominates $\hat{H}$ and $H(.|\mathcal{Y} \geq 0)$ reverse hazard rate dominates $\hat{H}(.|\mathcal{Y} \geq 0)$. This implies

$$\frac{h(.)}{H(.)} \geq \frac{\hat{h}(.)}{\hat{H}(.)}, \quad \frac{h(.|\mathcal{Y} \geq 0)}{H(.|\mathcal{Y} \geq 0)} \geq \frac{\hat{h}(.|\mathcal{Y} \geq 0)}{\hat{H}(.|\mathcal{Y} \geq 0)} \tag{18}$$

From Theorem 2, an equilibrium is

$$D\mu(v-t) = \frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_{v-t}Y(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)}[v-t-\mu(v-t)]$$

whenever the distribution pair is (F, G). If the distribution pair is $(\hat{F}, G)$, an equilibrium is

$$D\hat{\mu}(v-t) = \frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)\hat{q}(m)D_{v-t}\hat{Y}(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)\hat{q}(m)\hat{Y}(v-t,m)}[v-t-\hat{\mu}(v-t)]$$

where $\hat{Y}$ is the counterpart of Y. We show that

$$\frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_{v-t}Y(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)} \triangleq B(v-t) \geq$$
$$\frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)\hat{q}(m)D_{v-t}\hat{Y}(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)\hat{q}(m)\hat{Y}(v-t,m)} \triangleq \hat{B}(v-t)$$

For brevity, let

$$E(v-t,m) := m\frac{h(v-t|\mathcal{Y}\geq 0)}{H(v-t|\mathcal{Y}\geq 0)}$$

Similarly, $\hat{E}(.,m)$ is defined. Then,

$$B(v-t) = \frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)E(v-t,m)Y(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)}$$

Let

$$e(v-t,m) := \frac{\bar{Q}(m+1)q(m)Y(v-t,m)}{\sum_{m'\in\bar{N}} \bar{Q}(m'+1)q(m)Y(v-t,m')}$$

Similarly, $\hat{e}(.,m)$ is defined. Then,

$$B(v-t) = \sum_{m\in\bar{N}} e(v-t,m)E(v-t,m)$$

and

$$\hat{B}(v-t) = \sum_{m\in\bar{N}} \hat{e}(v-t,m)\hat{E}(v-t,m)$$

Note that $\sum_{m\in\bar{N}} e(v-t,m) = \sum_{m\in\bar{N}} \hat{e}(v-t,m) = 1$ for every $v,t\in[0,a]$ with $v\geq t$. Given $v,t\in[0,a]$ with $v\geq t$,

$$\begin{aligned}
B(v-t)-\hat{B}(v-t) = &\sum_{m\in\bar{N}} e(v-t,m)E(v-t,m) - \\
&\sum_{m\in\bar{N}} \hat{e}(v-t,m)\hat{E}(v-t,m) - \sum_{m\in\bar{N}} e(v-t,m)\hat{E}(v-t,m) \\
&+ \sum_{m\in\bar{N}} e(v-t,m)\hat{E}(v-t,m) \\
&= \sum_{m\in\bar{N}} e(v-t,m)[E(v-t,m)-\hat{E}(v-t,m)] + \\
&\quad \sum_{m\in\bar{N}} \hat{E}(v-t,m)[e(v-t,m)-\hat{e}(v-t,m)]
\end{aligned}$$

We show that $B(.)-\hat{B}(.)\geq 0$. From (18), we have $E(.,m)\geq\hat{E}(.,m)$. Therefore, it reduces to show that $\sum_{m\in\bar{N}} \hat{E}(.,m)[e(v-t,m)-\hat{e}(.,m)]\geq 0$.

We show that $\hat{E}(.,m+1) > \hat{E}(.,m)$ for every $m\in\bar{N}$. Consider $m\in\bar{N}$. Since

$$\hat{E}(.,m+1)-\hat{E}(.,m)=(m+1)\frac{\hat{h}(.|\mathcal{Y}\geq 0)}{\hat{H}(.|\mathcal{Y}\geq 0)}-m\frac{\hat{h}(.|\mathcal{Y}\geq 0)}{\hat{H}(.|\mathcal{Y}\geq 0)},$$

we have $\hat{E}(.,m+1)>\hat{E}(.,m)$.

We show that $e(.,m)/\hat{e}(.,m)$ is non-decreasing in m. Since

$$\frac{e(.,m)}{\hat{e}(.,m)}=\frac{q(m)H(.|\mathcal{Y}\geq 0)^m}{\hat{q}(m)\hat{H}(.|\mathcal{Y}\geq 0)^m}\frac{\sum_{m'\in\bar{N}}\bar{Q}(m'+1)\hat{q}(m')\hat{Y}(.,m')}{\sum_{m'\in\bar{N}}\bar{Q}(m'+1)q_i(m')Y(.,m')}$$

and

$$\frac{e(.,m+1)}{\hat{e}(.,m+1)}=\frac{q(m+1)H(.|\mathcal{Y}\geq 0)^{m+1}}{\hat{q}(m+1)\hat{H}(.|\mathcal{Y}\geq 0)^{m+1}}\frac{\sum_{m'\in\bar{N}}\bar{Q}(m'+1)\hat{q}(m')\hat{Y}(.,m')}{\sum_{m'\in\bar{N}}\bar{Q}(m'+1)q(m')Y(.,m')},$$

we have

$$\begin{aligned}\frac{e(.,m+1)/\hat{e}(.,m+1)}{e(.,m)/\hat{e}(.,m)}&=\frac{q(m+1)/\hat{q}(m+1)}{q(m)/\hat{q}(m)}\frac{H(.|\mathcal{Y}\geq 0)}{\hat{H}(.|\mathcal{Y}\geq 0)}\\&=\frac{[H(y)-H(0)]\hat{H}(0)}{[\hat{H}(y)-\hat{H}(0)]H(0)}\end{aligned}$$

Since $H(.)$ reverse hazard rate dominates $\hat{H}(.)$ (Lemma 3),[4] we have $e(.,m+1)/\hat{e}(.,m+1)\geq e(.,m)/\hat{e}(.,m)$. Hence, $e(.,m)/\hat{e}(.,m)$ is non-decreasing in m. We show that

$$\sum_{m=0}^{k}e(.,m)\leq\sum_{m=0}^{k}\hat{e}(.,m)$$

for every $k\leq n$. Consider $k\in\bar{N}$. Since $\sum_{m\in\bar{N}}e(.,m)=\sum_{m\in\bar{N}}\hat{e}(.,m)=1$, we have $\sum_{m\in\bar{N}}[e(.,m)-\hat{e}(.,m)]=0$. This implies

$$\sum_{m\in\bar{N}}\hat{e}(.,m)\left[\frac{e(.,m)}{\hat{e}(.,m)}-1\right]=0$$

Since $e(.,m)/\hat{e}(.,m)$ is non-decreasing and the above condition holds, there exists $\bar{m}$ so that $e(.,m)\leq\hat{e}(.,m)$ for every $m\leq\bar{m}$ and $e(.,m)\geq\hat{e}(.,m)$ for every $m>\bar{m}$. This implies $\sum_{m=0}^{k}e(.,m)\leq\sum_{m=0}^{k}\hat{e}(.,m)$ for every $k\leq\bar{m}$. Since

$$\sum_{m=0}^{k}e(.,m)=\sum_{m=0}^{n}e(.,m)-\sum_{m=k+1}^{n}e(.,m)=1-\sum_{m=k+1}^{n}e(.,m)$$

[4]If $H(.|\mathcal{Y}\geq 0)$ reverse hazard rate dominates $\hat{H}(.|\mathcal{Y}\geq 0)$, then $H(.|\mathcal{Y}\geq 0)/\hat{H}(.|\mathcal{Y}\geq 0)$ is non-decreasing.

and $\sum_{m=0}^{k} \hat{e}(.,m) = 1 - \sum_{m=k+1}^{n} \hat{e}(.,m)$, for every $k > m > \bar{m}$, we have $\sum_{m=0}^{k} e(.,m) \leq \sum_{m=0}^{k} \hat{e}(.,m)$. Therefore, $\sum_{m=0}^{k} e(.,m) \leq \sum_{m=0}^{k} \hat{e}(.,m)$ for every $k \in \bar{N}$.

We show that $\sum_{m\in\bar{N}} \hat{E}(.,m)[e(v-t,m) - \hat{e}(.,m)] \geq 0$. The left hand side can be written in expanded form as

$$
\begin{aligned}
&\sum_{m\in\bar{N}} \hat{E}(.,m)[e(.,m) - \hat{e}(.,m)] \\
&= \sum_{m\in\bar{N}} \hat{E}(.,m)\Big\{\sum_{i=0}^{m}[e(.,i) - \hat{e}(.,i)] - \sum_{i=0}^{m-1}[e(.,i) - \hat{e}(.,i)]\Big\} \\
&= \sum_{m\in\bar{N}} \sum_{i=0}^{m} \hat{E}(.,m)[e(.,i) - \hat{e}(.,i)] - \sum_{m\in\bar{N}} \sum_{i=0}^{m-1} \hat{E}(.,m)[e(.,i) - \hat{e}(.,i)]
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\sum_{m\in\bar{N}} \hat{E}(.,m)[e(.,m) - \hat{e}(.,m)] \\
&= \sum_{m\in\bar{N}} \sum_{i=0}^{m} \hat{E}(.,m)[e(.,i) - \hat{e}(.,i)] - \sum_{m=-1}^{n-1} \sum_{i=0}^{m} \hat{E}(.,m+1)[e(.,i) - \hat{e}(.,i)] \\
&= \sum_{i=0}^{n} \hat{E}(.,n)[e(.,i) - \hat{e}(.,i)] + \sum_{m=0}^{n-1} \sum_{i=0}^{m} \hat{E}(.,m)[e(.,i) - \hat{e}(.,i)] \\
&\quad - \sum_{i=0}^{-1} \hat{E}(.,0)[e(.,i) - \hat{e}(.,i)] - \sum_{m=0}^{n-1} \sum_{i=0}^{m} \hat{E}(.,m+1)[e(.,i) - \hat{e}(.,i)] \\
&= \sum_{m=0}^{n-1} \sum_{i=0}^{m} \hat{E}(.,m)[e(.,i) - \hat{e}(.,i)] - \sum_{m=0}^{n-1} \sum_{i=0}^{m} \hat{E}(.,m+1)[e(.,i) - \hat{e}(.,i)]
\end{aligned}
$$

as $\sum_{i=0}^{n} \hat{E}(.,n)[e(.,i) - \hat{e}(.,i)] = 0$. The above expression can be rewritten as

$$
\begin{aligned}
\sum_{m\in\bar{N}} \hat{E}(.,m)[e(.,m) - \hat{e}(.,m)] = &\sum_{m=0}^{n-1} \sum_{i=0}^{m} [e(.,i) - \\
&\hat{e}(.,i)][\hat{E}(.,m) - \hat{E}(.,m+1)] \\
\geq\ &0
\end{aligned}
$$

Hence $B(.) - \hat{B}(.) \geq 0$.

Recall that $D\mu(v-t) = B(v-t,m)[v-t-\mu(v-t)]$ and $D\hat{\mu}(v-t) = \hat{B}(v-t,m)[v-t-\hat{\mu}(v-t)]$. Note that $\mu(0) = \hat{\mu}(0) = 0$. If $\mu(v-t) < \hat{\mu}(v-t)$, then $D\mu(v-t) > D\hat{\mu}(v-t)$. Thus, $\mu(v-t,m) \geq \hat{\mu}(v-t,m)$ for every $v,t \in [0,a]$ with $v \geq t$. ■

Proof of Corollary 7. Consider that there is no floor. From Corollary 5, we have

$$D\mu^*(v-t) = n\frac{h(v-t)}{H(v-t)}[v-t-\mu^*(v-t)]$$

and

$$D\hat{\mu}^*(v-t) = n\frac{\hat{h}(v-t)}{\hat{H}(v-t)}[v-t-\hat{\mu}^*(v-t)]$$

From Lemma 3, we have

$$\frac{h(.)}{H(.)} \geq \frac{\hat{h}(.)}{\hat{H}(.)}$$

This implies that if $\mu^*(v-t) < \hat{\mu}^*(v-t)$, then $D\mu^*(v-t) > D\hat{\mu}^*(v-t)$. Therefore, the result holds. ■

Proof of Theorem 5. Consider $v, t \in [0, a]$ with $v > t$. We show $\sigma(v,t) > \sigma^*(v,t)$. From Theorem 2, we have

$$D\mu(v-t) = \frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_{v-t}Y(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)}[v-t-\mu(v-t)]$$

The first-order condition of Corollary 5 is rewritten as

$$D\mu^*(v-t) = \frac{\sum_{k\in\bar{N}} q(k)D_{v-t}Y(v-t,k)}{\sum_{k\in\bar{N}} q(k)Y(v-t,k)}[v-t-\mu^*(v-t)]$$

We show that

$$\frac{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)D_{v-t}Y(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)} \geq \frac{\sum_{k\in\bar{N}} q(k)D_{v-t}Y(v-t,k)}{\sum_{k\in\bar{N}} q(k)Y(v-t,k)}$$

which is equivalent to

$$\frac{\sum_{m\in\bar{N}} m\bar{Q}(m+1)q(m)Y(v-t,m)}{\sum_{m\in\bar{N}} \bar{Q}(m+1)q(m)Y(v-t,m)} \geq \frac{\sum_{k\in\bar{N}} kq(k)Y(v-t,k)}{\sum_{k\in\bar{N}} q(k)Y(v-t,k)}$$

This reduces to

$$\sum_{m\in\bar{N}}\sum_{k\in\bar{N}} \bar{Q}(m+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(m-k) \geq 0$$

Interchanging the indices, we have

$$\sum_{m\in\bar{N}}\sum_{k\in\bar{N}}\bar{Q}(m+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(k-m)\geq 0$$

Summing the above two equations, we have

$$\sum_{m\in\bar{N}}\sum_{k\in\bar{N}}q(m)q(k)Y(v-t,m)Y(v-t,k)(m-k)[\bar{Q}(m+1)-\bar{Q}(k+1)]\geq 0$$

Since $\bar{Q}(.)$ is non-decreasing, the above condition holds.

If $\mu(v-t)<\mu^*(v-t)$, then $D_{v-t}\mu(v-t)>D_{v-t}\mu^*(v-t)$. Since $\mu(0)=\mu^*(0)=0$, $\mu(v-t)\geq\mu^*(v-t)$ for every $v,t\in[0,a]$ with $v\geq t$.

We show $\sigma(v,t,m^*)>\sigma^*(v,t)$. From Corollaries 2 and 5, we have

$$D\mu(v-t,m^*)=\frac{\sum_{k=m^*}^{n}q(k)D_{v-t}Y(v-t,k)}{\sum_{k=m^*}^{n}q(k)Y(v-t,k)}[v-t-\mu(v-t,m^*)]$$

and

$$D\mu^*(v-t)=\frac{\sum_{m\in\bar{N}}q(m)D_{v-t}Y(v-t,m)}{\sum_{m\in\bar{N}}q(m)Y(v-t,m)}[v-t-\mu^*(v-t)]$$

We show

$$\frac{\sum_{k=m^*}^{n}q(k)D_{v-t}Y(v-t,k)}{\sum_{k=m^*}^{n}q(k)Y(v-t,k)}>\frac{\sum_{m\in\bar{N}}q(m)D_{v-t}Y(v-t,m)}{\sum_{m\in\bar{N}}q(m)Y(v-t,m)}$$

which is equivalent to

$$\frac{\sum_{k=m^*}^{n}kq(k)Y(v-t,k)}{\sum_{k=m^*}^{n}q(k)Y(v-t,k)}>\frac{\sum_{m\in\bar{N}}mq(m)Y(v-t,m)}{\sum_{m\in\bar{N}}q(m)Y(v-t,m)}$$

This reduces to

$$\sum_{k=m^*}^{n}\sum_{m\in\bar{N}}q(k)q(m)Y(v-t,k)Y(v-t,m)(k-m)>0$$

which is equivalent to

$$\begin{aligned}0<\sum_{k=m^*}^{n}\sum_{m=0}^{m^*-1}&q(k)q(m)Y(v-t,k)Y(v-t,m)(k-m)+\\ \sum_{k=m^*}^{n}\sum_{m=m^*}^{n}&q(k)q(m)Y(v-t,k)Y(v-t,m)(k-m)\end{aligned}$$

Interchanging the indices gives

$$0<\sum_{m=m^*}^{n}\sum_{k=0}^{m^*-1}q(k)q(m)Y(v-t,k)Y(v-t,m)(m-k)+$$
$$\sum_{k=m^*}^{n}\sum_{m=m^*}^{n}q(k)q(m)Y(v-t,k)Y(v-t,m)(m-k)$$

Summing the above two inequalities, we have

$$\sum_{k=m^*}^{n}\sum_{m=0}^{m^*-1}q(k)q(m)Y(v-t,k)Y(v-t,m)(k-m)>0$$

Since $k>m$, the above inequality holds. The rest of the proof is analogous to the first part. ■

Proof of Theorem 6. We show the first part. Given $v,t\in[0,a]$ with $v>t$, let

$$C(v-t):=\frac{h(v-t|\mathcal{Y}\geq 0)}{H(v-t|\mathcal{Y}\geq 0)}\frac{\sum_{m\in\bar{N}}m\bar{Q}(m+1)q(m)Y(v-t,m)}{\sum_{m\in\bar{N}}\bar{Q}(m+1)q(m)Y(v-t,m)}$$

and

$$C(v-t,m^*):=\frac{h(v-t|\mathcal{Y}\geq 0)}{H(v-t|\mathcal{Y}\geq 0)}\frac{\sum_{k=m^*}^{n}kq(k)Y(v-t,k)}{\sum_{k=m^*}^{n}q(k)Y(v-t,k)}$$

Since $\bar{q}(m)=0$ for every $m<m^*+1$, $\bar{Q}(m)=0$ for every $m<m^*+1$. Consequently,

$$C(v-t):=\frac{h(v-t|\mathcal{Y}\geq 0)}{H(v-t|\mathcal{Y}\geq 0)}\frac{\sum_{m=m^*}^{n}m\bar{Q}(m+1)q(m)Y(v-t,m)}{\sum_{m=m^*}^{n}\bar{Q}(m+1)q(m)Y(v-t,m)}$$

We show $C(.)\geq C(.,m^*)$ which is equivalent to

$$\sum_{m=m^*}^{n}\sum_{k=m^*}^{n}\bar{Q}(m+1)q(m)q(k)Y(.,m)Y(.,k)(m-k)\geq 0$$

Interchanging the indices, we have

$$\sum_{m=m^*}^{n}\sum_{k=m^*}^{n}\bar{Q}(k+1)q(k)q(m)Y(.,k)Y(.,m)(k-m)\geq 0$$

Summing the above two equations, we have

$$\sum_{m=m^*}^{n}\sum_{k=m^*}^{n} q(m)q(k)Y(.,m)Y(.,k)(m-k)[\bar{Q}(m+1)-\bar{Q}(k+1)] \geq 0$$

Since $\bar{Q}$ is non-decreasing, the above condition holds.

If $\mu(v-t) < \mu(v-t,m^*)$, from Theorem 2 and Corollary 3, we have $D_{v-t}\mu(v-t) > D_{v-t}\mu(v-t,m^*)$. Since $\mu(0) = \mu(0,m^*)$, we have $\mu(v-t) \geq \mu(v-t,m^*)$ for every $v,t \in [0,a]$ with $v > t$.

We show the second part. Since $\bar{q}(m) = 0$ for every $m > m^*+1$, $\bar{Q}(m) = 1$ for every $m \geq m^*+1$. As a result,

$$C(v-t,m^*) = \frac{h(v-t|\mathcal{Y}\geq 0)}{H(v-t|\mathcal{Y}\geq 0)} \frac{\sum_{k=m^*}^{n} k\bar{Q}(k+1)q(k)Y(v-t,k)}{\sum_{k=m^*}^{n} \bar{Q}(k+1)q(k)Y(v-t,k)}$$

We show $C(v-t,m^*) > C(v-t)$ which is equivalent to

$$\sum_{k=m^*}^{n}\sum_{m\in\bar{N}} \bar{Q}(m+1)\bar{Q}(k+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(k-m) > 0$$

It expands to

$$\begin{aligned}0 <& \sum_{k=m^*}^{n}\sum_{m=0}^{m^*-1} \bar{Q}(m+1)\bar{Q}(k+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(k-m)\\ &+ \sum_{k=m^*}^{n}\sum_{m=m^*}^{n} \bar{Q}(m+1)\bar{Q}(k+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(k-m)\end{aligned}$$

Interchanging the indices, we have

$$\begin{aligned}0 <& \sum_{m=m^*}^{n}\sum_{k=0}^{m^*-1} \bar{Q}(m+1)\bar{Q}(k+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(m-k)\\ &+ \sum_{k=m^*}^{n}\sum_{m=m^*}^{n} \bar{Q}(m+1)\bar{Q}(k+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(m-k)\end{aligned}$$

Summing the above two inequalities, we have

$$\sum_{k=m^*}^{n}\sum_{m=0}^{m^*-1} \bar{Q}(m+1)\bar{Q}(k+1)q(m)q(k)Y(v-t,m)Y(v-t,k)(k-m) > 0$$

Since $k > m$, the above inequality is true. The rest of the proof is similar to the first part. ∎

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