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Stable Matchings with Choice Correspondences Under Acyclicity

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Abstract

We study the existence of stable matchings when agents have choice correspondences instead of preference relations. We extend the framework of [Chambers and Yenmez \(2017\)](#) by weakening the Path Independence assumption. For many-to-many markets, we show that stable matchings exist when choice correspondences satisfy Substitutability and a new General Acyclicity condition. We provide a constructive proof using a Grow or Discard Algorithm that iteratively expands or eliminates contracts until a strongly maximal Individually Rational set is reached. We provide an algorithm to obtain stable matchings in which rejected contracts are not permanently discarded, distinguishing our approach significantly from standard DAA-type algorithms. For one-to-one markets, we introduce a replacement-based notion of stability and provide an algorithm that constructs stable matchings when choice correspondences satisfy Binary Acyclicity, a property weaker than Path Independence.

JEL classification: C62, C78, D01, D47

Keywords: choice correspondences, substitutability, general acyclicity, many-to-many matching, matching with contracts, Grow or Discard algorithm, replacement stability, binary acyclicity.

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1 Introduction

The matching literature typically models agents as having complete and transitive preference orderings over feasible outcomes from which stable matchings are derived. While this framework is intuitive and successful, it relies on strong informational and behavioral assumptions. In many settings, the preferences of agents are not known. Agents may also exhibit bounded rationality or non-standard choice behavior that prevents their choices from being represented by strict rankings.¹

In this paper, we depart from the standard approach by studying matching markets in which agents are described entirely by choice correspondences. The primitives of our model are observable choices agents make from feasible sets of contracts. This formulation accommodates non-standard behaviors that strict preference rankings rule out, such as context-dependent choices where an agent might desire a previously rejected contract only when a new, complementary contract is offered alongside it.

It is well known that additional consistency conditions on choices are required to guarantee the existence of stable matchings in such environments (Aygün and Sönmez (2013) and Hatfield and Milgrom (2005)). Chambers and Yenmez (2017) (henceforth CY) show that when choice correspondences satisfy Path Independence, stable matchings exist in many-to-many markets. Path independence, however, is a strong restriction: it requires that choices from a set coincide with those obtained through any sequential partition of the set. Our goal is to identify weaker behavioral conditions that still guarantee the existence of stable matchings. We provide the informal statements of our main results which show that weaker conditions indeed suffice.

Main Results (informal).

- In many-to-many markets, if choice correspondences satisfy Substitutability and General Acyclicity, then a CY-stable matching exists which can be computed by an algorithm which is significantly different from a DAA-type algorithm. We show that these conditions are strictly weaker than assuming Path Independence.²
- In one-to-one markets, we introduce a new notion of stability in terms of replacement and we show that a property weaker than Path Independence, i.e. Binary Acyclicity, is sufficient to guarantee Replacement Stability.

¹In particular, their choices may fail to be representable by Complete and Transitive binary relations. A binary relation R over X is (i) Complete if for all $x, y \in X$, either xRy or yRx (or both), and (ii) Transitive if for any $x, y, z \in X$, $[xRy \text{ and } yRz] \implies [xRz]$.

²In the many-to-many problem, we define stability as in CY. We call it CY-stability.

Most papers that study matching with choice correspondences assume some form of substitutability. While Substitutability is sufficient to guarantee stability under standard preference assumptions in the one-to-one matching problem (Gale and Shapley (1962)), in many-to-many matching environments, the existence of stable outcomes requires additional structure on agents' choices. In particular, Hatfield and Milgrom (2005) show that when agents' choice functions satisfy Substitutability along with Irrelevance of Rejected Contracts, stable allocations exist in general matching markets with contracts.

In the many-to-many matching problem, we show the existence of a CY-stable matching when choice correspondences satisfy Substitutability (SUB) and General Acyclicity (GA). To prove this, we first show the existence of a strongly maximal 'stable' set in the individual choice problem.³ Starting from an individually rational set of contracts, the algorithm iteratively considers challenger sets (any non-empty disjoint set) that may improve upon the current matching. If a challenger set is weakly preferred, the allocation grows to add that set. On the other hand, if it is strictly preferred, some previously chosen contracts are discarded and the set moves to the new choice.

We show that our matching algorithm, the Dynamic Grow or Discard Matching Algorithm (GDMA) can be implemented where agents on one side of the market propose contracts for the side to accept or reject. Agents on the other side again use the 'grow or discard' algorithm for individual choices to decide whether to 'grow' by expanding the set or to 'discard' some old contracts and move to a different set of contracts. Unlike standard deferred acceptance algorithms, the Dynamic GDMA allows firms to re-propose previously rejected contracts. Proving the stability of the obtained matching, therefore, requires new arguments than the ones used in the literature.

Substitutability ensures that individual rationality is preserved at every step. General acyclicity rules out cycles of grow or discard moves and guarantees that the algorithm does not loop indefinitely. Since the set of feasible contracts is finite, the process terminates at a set which is individually rational which cannot 'grow' further. Therefore, the GDMA relies on acyclicity in revealed choices to converge to a CY-stable matching.

We then provide results for the one-to-one matching problem, which is not studied in CY. We introduce a different notion of stability based on the replacement of current matches when choices are acyclic over binary sets. Our stability notion captures the idea that a matching remains stable not because agents possess a complete ranking, but because no

³A strongly maximal stable set of contracts for an agent is a set of contracts that the agent would choose to retain, and for which no additional set of contracts from outside the set would be chosen when available.

agent can find an alternative partner who would simultaneously choose to replace their current match. We provide a review of the literature surrounding this work to place our contribution.

1.1 Literature Review

The classical theory of stable matchings begins with [Gale and Shapley \(1962\)](#), who study one-to-one matching markets with strict preferences and show that stable matchings exist and form a lattice. Subsequent work extended these results to environments in which agents have preferences over sets of partners. In particular, [Kelso Jr and Crawford \(1982\)](#) introduce substitutability in job matching markets with wages, and [Roth \(1984\)](#) show that stable matchings exist in many-to-one markets when firms' preferences are substitutable.⁴

A major generalization is provided by [Hatfield and Milgrom \(2005\)](#) which studies matching markets (many-to-one) in terms of more general contracts where agents have choice functions and stability in terms of the latter. They show that Substitutability alone guarantees the existence of stable allocations. [Hatfield and Kojima \(2010\)](#) weakened the condition in [Hatfield and Milgrom \(2005\)](#) by constraining the range of complementarities (unilateral substitutability), while [Hatfield and Kominers \(2017\)](#) consider choice functions that can be extended to a fully substitutable preference structure (completability).⁵ [Aygün and Sönmez \(2013\)](#) clarified the results established by [Hatfield and Milgrom \(2005\)](#) by showing that Substitutability alone is not sufficient to guarantee the existence of stable matchings and demonstrate that the additional condition of Irrelevance of Rejected Contracts (IRC) is required.

[Alkan and Gale \(2003\)](#) study multi-valued choice in matching and extend the results of [Alkan \(2001\)](#) and [Alkan \(2002\)](#) to show that stable matchings exist and that the set of stable matchings forms a lattice. They provide a choice-theoretic foundation for stability in many-to-many matching markets, showing that stable matchings exist and form a lattice under additional conditions without requiring preference orderings.⁶

While the literature following [Hatfield and Milgrom \(2005\)](#) and [Aygün and Sönmez \(2013\)](#)

⁴Building on this line of work, [Blair \(1988\)](#) studies the structure of the set of stable matchings and shows that, under appropriate conditions, stable matchings form a lattice.

⁵Completability requires that as long as the “badly behaved” (complementary) preferences can be encompassed inside a larger structure of “well behaved” (substitutable) choice domain, a stable matching is guaranteed to exist.

⁶Their results rely on two conditions on choice behavior: (i) Persistence, which requires that if an alternative is rejected from a set it remains rejected when additional alternatives become available, and (ii) Size Monotonicity, which requires that the number of chosen alternatives does not decrease when the set of available alternatives expands.

has successfully identified several sufficient conditions for stability, these restrictions often demand strong conditions on the choice behavior. The present paper bridges this gap by exploring the boundary of these constraints, identifying weaker, acyclicity-type behavioral requirements that suffice for existence in finite contract markets. We discuss the contributions in the next subsection.

1.2 Contribution to the Literature

Chambers and Yenmez (2017) establish existence under Path Independence, a strong restriction that can be decomposed into Substitutability combined with an expansion-type consistency axiom. The present paper is closely related in spirit to this paper: we relax the requirement of Path Independence on choice correspondences to study matching markets with choice correspondences over a finite set of contracts. Our results show that the existence of stable matchings can be preserved under significantly weaker, acyclicity-type conditions.

To establish existence, provide a constructive proof via a novel ‘Grow or Discard’ Algorithm (GDA), which iteratively builds individually rational sets and terminates in finite time with a strongly maximal Individually Rational set. We show that our matching algorithm, the Dynamic Grow or Discard Matching Algorithm (GDMA)), unlike standard deferred acceptance algorithms, allows firms to re-propose previously rejected contracts. However, the algorithm strictly requires that any transition to a new tentative matching must be strictly validating for both the proposing firm(s) and the accepting workers. Since acyclicity condition strictly rules out revealed preference cycles in the transition path, the GDMA travels through an acyclic path of matchings. Therefore, the process converges in finite time, terminating in a state with no further choice improvements.

Our results imply that failures of stability cannot arise from violations of Substitutability per se. The cyclic of revealed choices across different sets of contracts can cause problems. This perspective helps reconcile several strands of the literature that impose stronger assumptions, such as Irrelevance of Rejected Contracts or Path Independence. Our framework provides the essential regularity needed to move beyond existence results and opens the door for a systematic study of comparative statics in matching markets with general choice correspondences

The paper is organized as follows. Section 2 introduces the model, defines the matching environments, while Section 3 presents the axioms on choice correspondences used in the paper. Section 4 analyzes the many-to-many matching problem and presents the Grow or Discard Algorithm and discusses its properties. Section 5 studies the one-to-one

matching problem, introduces the notion of stability, and establishes existence results with a constructive algorithm. The list of references are provided at the end. All proofs not provided in the main text are provided in the Appendix.

2 Model

There is a finite set of workers \mathbf{W} , firms \mathbf{F} , and contracts \mathbf{X} . Each contract $x \in \mathbf{X}$ identifies a unique firm-worker pair with a comprehensive set of employment terms, which may include wages and other contractual benefits. For each contract $x \in \mathbf{X}$, the firm and worker associated with the contract are denoted by $f(x)$ and $w(x)$, respectively.⁷

For a set of contracts $X \subseteq \mathbf{X}$, let $X_w = \{x \in X : w = w(x)\}$ be the set of contracts associated with worker w and let $X_f = \{x \in X : f = f(x)\}$ be the set of contracts associated with firm f . Let \mathbf{X}_w and \mathbf{X}_f be the set of all contracts associated with worker w and firm f respectively. Let \mathcal{X} be the set of all subsets of \mathbf{X} , let \mathcal{X}_w be the set of all subsets of \mathbf{X}_w and \mathcal{X}_f the set of all subset of \mathbf{X}_f .

Each worker w has a choice rule $C_w : \mathcal{X}_w \rightarrow \mathcal{X}_w$ such that $C_w(S) \subseteq S$ for all $S \in \mathcal{X}_w$. Note that we allow choices to be empty sets since \mathcal{X}_w contains the empty set \emptyset . Similarly, each firm f has a choice function $C_f : \mathcal{X}_f \rightarrow \mathcal{X}_f$ such that $C_f(S) \subseteq S$ for all $S \in \mathcal{X}_f$. We assume that an agent's choice depends only on the contracts available to them, that is, for any $S \subseteq X$, $C_w(S) = C_w(S \cap X_w)$ and $C_f(S) = C_f(S \cap X_f)$. A matching market is a tuple $\langle \mathbf{W}, \mathbf{F}, \mathbf{X}, (C_w)_{w \in \mathbf{W}}, (C_f)_{f \in \mathbf{F}} \rangle$.

Definition 1 (Matching). A *matching* $\nu \in \mathcal{X}$ is a set of contracts. For each worker $w \in \mathbf{W}$, let $\nu_w = \{x \in \nu : w(x) = w\}$ denote the set of contracts assigned to w under ν . For each firm $f \in \mathbf{F}$, let $\nu_f = \{x \in \nu : f(x) = f\}$ denote the set of contracts assigned to f .

By default, a matching is many-to-many, i.e., multiple firms can be matched with multiple workers through any set of contracts. We assume there are no capacity constraints, i.e., each firm f can be matched to at most $|\mathbf{X}_f|$ number of contracts and each worker w can be matched to at most $|\mathbf{X}_w|$ set of contracts.

⁷Therefore, we allow for a bilateral contract to be formed between a worker and a firm. However, we do not put restrictions that they be unique, i.e., it is possible to have $x \neq x'$ such that $f(x) = f(x')$ and $w(x) = w(x')$.

3 Axioms

We provide some axioms for choice correspondences $C : \mathcal{X} \rightarrow \mathcal{X}$ where \mathcal{X} is the set of all subsets of X and $C(A) \subseteq A$ for all $A \in \mathcal{X}$.⁸ The first one is a consistency requirement which allows for substitutability between workers (or firms).

Definition 2 (Substitutability (SUB)). A choice correspondence C satisfies SUB if for every $A, B \subseteq X$ such that $A \subseteq B$, then $C(B) \cap A \subseteq C(A)$.

This axiom is the standard formulation of Substitutability in the matching with contracts literature, initially adapted from classical matching models by [Hatfield and Milgrom \(2005\)](#). It ensures that contracts are gross substitutes: if an agent chooses a contract from a larger set of available options, they will continue to choose that contract from any subset containing it.⁹ An alternative version of the axiom is stated as follows: a choice correspondence $C : \mathcal{X} \rightarrow \mathcal{X}$ is Contraction Consistent (CC) or SUB if for every $A, B \subseteq X$ such that $a \in A \subseteq B$, $[a \in C(B)] \Rightarrow [a \in C(A)]$. In other words, every alternative which was chosen from the superset and is also available in the subset should also be chosen in the subset. It is easy to verify that two definitions of SUB are equivalent. The following axiom, Consistency (CON), requires that if the chosen set from a set is also available from a subset, the choice remains unchanged when the menu is restricted to that subset.

Definition 3 (Consistency (CON)). A choice correspondence C satisfies CON if for every $A, B \subseteq X$ if $C(B) \subseteq A \subseteq B$, then $C(A) = C(B)$.

The next axiom is equivalent to the two axioms combined: SUB and CON.¹⁰ It is the central condition utilized by CY to guarantee the existence of a stable matching within the framework of choice correspondences. This condition requires that the choice from the union of two sets, $A \cup B$, be invariant to a two-stage selection process: first, choosing from each set independently, and subsequently choosing from the union of those preliminary selections. We provide the definition.

Definition 4 (Path Independence (PI)). A choice correspondence C is PI if for every $A, B \subseteq X$, $C(A \cup B) = C(C(A) \cup C(B))$.

⁸Note that \mathcal{X} can be replaced with \mathcal{X}_f or \mathcal{X}_w in our setting.

⁹[Kelso Jr and Crawford \(1982\)](#) introduced the gross substitutes condition in a job matching model with wages and preferences over sets of contracts to establish the existence of stable outcomes. [Roth \(1984\)](#) showed that stable many-to-one matchings exist under responsive preferences, which imply SUB. Subsequent work has adopted SUB directly as a condition on choice functions to guarantee stability, including [Hatfield and Milgrom \(2005\)](#), [Hatfield and Kojima \(2010\)](#), [Chambers and Yenmez \(2017\)](#), and [Pycia and Yenmez \(2023\)](#).

¹⁰[PI] \implies [SUB + CON] as shown in [Aizerman and Malishevski \(1981\)](#).

In the following section, we depart from the strict requirement of PI. We investigate the many-to-many matching model under a weaker behavioral assumption: we retain SUB to rule out complementarities but completely relax CON. We show that by replacing CON with our General Acyclicity (GA) condition, and applying a radically different matching algorithm (Dynamic Grow or Discard Matching Algorithm), we can still guarantee the existence of a stable matching.

4 The many-to-many matching problem

For worker w , a set of contracts $X \subseteq \mathcal{X}_w$ is revealed preferred to a set of contracts $Y \subseteq \mathcal{X}_w$ if $C_w(X \cup Y) = X$. Symmetrically, for firm f , a set of contracts $X \subseteq \mathcal{X}_f$ is revealed preferred to a set of contracts $Y \subseteq \mathcal{X}_f$ if $C_f(X \cup Y) = X$. This definition reflects that the revealed preference in this context is over sets of contracts, rather than individual contracts. We use the same notion of stability as the one used in [Chambers and Yenmez \(2017\)](#). We refer to this notion of stability as CY-stability.

Definition 5 (CY-Stability). A matching $\nu \in \mathcal{X}$ is CY-stable if it satisfies the following two properties:

1. **Individual rationality (IR).** A matching ν is IR if for every $f \in \mathbf{F}$ and $w \in \mathbf{W}$, $C_f(\nu_f) = \nu_f$ and $C_w(\nu_w) = \nu_w$.
2. **No CY-blocking:** A matching ν has No CY-blocking if there does not exist a non-empty set of contracts $X \subseteq \mathbf{X}$, with $X \cap \nu = \emptyset$, such that:
 - (i) For every firm $f \in \mathbf{F}$, $X_f \subseteq C_f(\nu_f \cup X_f)$, and
 - (ii) For every worker $w \in \mathbf{W}$, $X_w \subseteq C_w(\nu_w \cup X_w)$.

A matching ν is IR if every agent wants to keep all the contracts provided to her in the matching. A matching ν has No CY-blocking if there is no non-empty set of new contracts X (disjoint from ν) such that, when each firm and worker is offered its current contracts together with its contracts in X , every agent would choose to accept all of its contracts in X . In other words, there is no collection of contracts that all involved firms and workers would jointly accept in addition to their current match.

A many-to-many matching is CY-stable if it satisfies both these conditions. However, CY-stable matchings need not exist. CY identifies PI as a sufficient condition for the existence of CY-stable matchings. Our next example shows that this condition is not necessary and can be weakened further. Our subsequent analysis attempts to weaken

this condition significantly. For simplicity in illustration, we restrict the contracts to be the set of all firms (for workers) and the set of all workers (for firms).

Example 1. We show that PI of choice correspondences is not necessary for existence of CY-stable many-to-many matchings. Let $\mathbf{F} = \{f_1, f_2, f_3\}$ and $\mathbf{W} = \{w_1, w_2, w_3\}$. To simplify the example, we again assume that $\mathbf{X}_f = \{w_1, w_2, w_3\}$ for each $f \in \mathbf{F}$ and $\mathbf{X}_w = \{f_1, f_2, f_3\}$ for each $w \in \mathbf{W}$. Let $C_w(f_j) = \{f_j\}$ for every worker $w \in \mathbf{W}$ and $C_f(w_i) = \{w_i\}$ for every firm $f \in \mathbf{F}$. The other choice functions over other sets are given in the table below.¹¹

	Firms		
Contracts	f_1	f_2	f_3
$\{w_1, w_2\}$	$\{w_1\}$	$\{w_1\}$	$\{w_2\}$
$\{w_1, w_3\}$	$\{w_1, w_3\}$	$\{w_1\}$	$\{w_3\}$
$\{w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_2, w_3\}$
$\{w_1, w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_3\}$

	Workers		
Contracts	w_1	w_2	w_3
$\{f_1, f_2\}$	$\{f_1, f_2\}$	$\{f_1, f_2\}$	$\{f_1\}$
$\{f_1, f_3\}$	$\{f_1, f_3\}$	$\{f_1, f_3\}$	$\{f_1, f_3\}$
$\{f_2, f_3\}$	$\{f_3\}$	$\{f_2\}$	$\{f_2, f_3\}$
$\{f_1, f_2, f_3\}$	$\{f_1, f_2, f_3\}$	$\{f_1, f_2, f_3\}$	$\{f_1, f_2, f_3\}$

(a) Firms
(b) Workers

Table 1: Preferences for Example 1

Firms	ν_f	Workers	ν_w
f_1	$\{w_2, w_3\}$	w_1	$\{f_3\}$
f_2	$\{w_2, w_3\}$	w_2	$\{f_1, f_2, f_3\}$
f_3	$\{w_1, w_2, w_3\}$	w_3	$\{f_1, f_2, f_3\}$

Table 2: A CY-stable matching for Example 1

Firm f_1 's choices do not satisfy PI since $C_{f_1}(w_1, w_2) = \{w_1\}$, $C_{f_1}(w_3) = \{w_3\}$ but $C_{f_1}(w_1, w_2, w_3) = \{w_2, w_3\} \neq C_{f_1}(C_{f_1}(w_1, w_2) \cup C_{f_1}(w_3)) = C_{f_1}(w_1, w_3) = \{w_1, w_3\}$. Firm f_1 's choice does not satisfy PI. Let matching ν be: f_1 matched to w_1 and f_2 matched to w_2 and w_3 . Then, $\nu(w_1) = \{f_3\}$, $\nu(w_2) = \{f_1, f_2, f_3\}$, and $\nu(w_3) = \{f_1, f_2, f_3\}$. We leave it to the reader to verify that this matching is CY-stable.

We introduce another axiom, General Acyclicity (GA), which together with SUB, guarantees the existence of CY-stable many-to-many matchings. Lemma 1 implies that imposing SUB and GA together is still strictly weaker than assuming PI alone. We provide the formal definition below.

Definition 6 (General Acyclicity). A choice correspondence $C : \mathcal{X} \rightarrow \mathcal{X}$ satisfies General Acyclicity (GA) if for any non-empty sets $\{S_i\}_1^k$ and (possibly empty) sets $\{D_j\}_1^{k-1}$ in \mathcal{X} such that,

¹¹For simplicity of notation, we write $C(a, b)$ instead of $C(\{a, b\})$ and similar simplifications.

$$\left[C(S_1 \cup S_2) = S_2 \cup D_1, C((S_2 \cup D_1) \cup S_3) = S_3 \cup D_2, \dots, \text{ and } C((S_{k-2} \cup D_{k-3}) \cup S_{k-1}) = S_{k-1} \cup D_{k-2} \right]$$

$$\implies \left[S_1 \not\subseteq C((S_{k-1} \cup D_{k-2}) \cup S_k) \right],$$

where $D_1 \subsetneq S_1$, $D_i \subsetneq (S_i \cup D_{i-1})$ for $i \in \{2, \dots, k-2\}$ and $(D_{i-2} \cup S_{i-1}) \cap S_i = \emptyset$ for all $i \in \{3, \dots, k-1\}$.

We can use the notion of *revealed preference* through choice correspondence as follows: A is *revealed preferred* to B if $A \subseteq C(A \cup B)$. GA requires that revealed preferred cycles of the following kind do not occur: S_2 is revealed preferred to S_1 , S_3 is revealed preferred to the choice made in the previous stage, (i.e. $S_2 \cup D_1$), ..., S_{k-1} is revealed preferred to $(S_{k-2} \cup D_{k-3})$ and finally S_1 through S_k is revealed preferred (to $S_{k-1} \cup D_{k-1}$).¹² In other words, choice over pairwise disjoint sets should not ‘cycle’ in a way where the initial set is discarded and then again selected (as a subset possibly) at the end of the cycle.

Note that for $k = 2$, GA holds vacuously. A violation of GA for $k = 2$ would require that $C(S_1 \cup S_2) = S_2 \cup D_1$ where $D_1 \subsetneq S_1$, and $C((S_2 \cup D_1) \cup S_1) = S_1 \cup D_2$ where $D_2 \subsetneq S_2 \cup D_1$. However, this implies that $(S_1 \cup S_2) \subseteq C(S_1 \cup S_2)$. This along with the fact that $D_1 \subsetneq S_1$ is a contradiction. Therefore, GA prevents cycles of length three and above from occurring in a ‘grow’ or ‘discard’ manner. At each step starting from the first: where we start with the set S_1 move to another set by adding S_2 and discarding some elements from S_1 , then moving to S_3 and discarding some elements from $S_2 \cup D_1$, and so on, ending with the last stage where we again obtain the whole set S_1 which was (partially) ‘discarded’ before reaching the last stage. We now show that PI implies GA.

Lemma 1. Suppose choice correspondence $C : \mathcal{X} \rightarrow \mathcal{X}$ satisfies PI. Then C satisfies GA.

Proof. Suppose the choice correspondence C satisfies PI but not GA. Then there exist non-empty sets $\{S_j\}_1^k$ and (possibly empty) sets $\{D_j\}_1^{k-1}$ in \mathcal{X} such that,

$$C(S_1 \cup S_2) = S_2 \cup D_1, C((S_2 \cup D_1) \cup S_3) = S_3 \cup D_2, \dots,$$

$$C((S_{k-2} \cup D_{k-3}) \cup S_{k-1}) = S_{k-1} \cup D_{k-2}, \text{ and } C((S_{k-1} \cup D_{k-2}) \cup S_k) = S_1 \cup D_{k-1},$$

¹²Note that no relationship is assumed between S_1 and S_k .

where $D_1 \subsetneq S_1$, $D_i \subsetneq (S_i \cup D_{i-1})$ for $i \in \{2, \dots, k-1\}$ and $(D_{i-2} \cup S_{i-1}) \cap S_i = \emptyset$ for all $i \in \{3, \dots, k\}$. Let $A = S_1 \cup S_2 \cup \dots \cup S_k$.

Since PI implies SUB¹³, and SUB is equivalent to CC, we can use the contrapositive of CC: Given a choice function $C : \mathcal{X} \rightarrow \mathcal{X}$ the contrapositive of CC is as follows: for any $A \subseteq B \in \mathcal{X}$, $[x \notin C(A)] \implies [x \notin C(B)]$. By the contrapositive of CC,

$$[S_1 \not\subseteq C(S_1 \cup S_2)] \implies [S_1 \not\subseteq C(A)].$$

Therefore, there exists $x \in S_1$ such that $x \notin C(S_1 \cup S_2)$. By SUB, $[x \notin C(S_1 \cup S_2)] \implies [x \notin C(S_1 \cup S_2 \cup S_k \cup D_{k-1})]$. Recall that PI \implies CON (Aizerman and Malishevski (1981)). Therefore, by CON, $[x \notin C(S_1 \cup S_2 \cup S_k \cup D_{k-1})] \implies [x \notin C((S_k \cup D_{k-1}) \cup S_{k+1})]$. This is a contradiction to the fact that $S_1 \subseteq C((S_k \cup D_{k-1}) \cup S_{k+1})$ as assumed. Therefore, PI \implies GA. ■

Next, we provide example of a choice correspondence that satisfies SUB and GA but not PI.

Example 2. Let $X = \{a, b, c, d\}$. We provide an example of a choice function which satisfies SUB and GA but not PI. Let $C(x) = \{x\}$ for all $x \in X$. The other choices are given in the table below.

Subset(s)	$C(\cdot)$
X	$\{a, d\}$
$\{a, b\}$	$\{a, b\}$
$\{a, c\}$	$\{a\}$
$\{a, d\}$	$\{a, d\}$
$\{b, c\}$	$\{b\}$
$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$
$\{a, b, c\}$	$\{a, b\}$
$\{a, b, d\}$	$\{a, b, d\}$
$\{a, c, d\}$	$\{a, d\}$
$\{b, c, d\}$	$\{b, d\}$

Table 3: Choices which satisfy SUB and GA but not PI

The choice function is not PI since $C(X) = \{a, d\}$, but $C(C(a, b, c) \cup C(d)) = C(a, b, d) =$

¹³Recall that Aizerman and Malishevski (1981) proved that PI implies SUB.

$\{a, b, d\}$ while PI requires that $C(X) = C(C(a, b, c) \cup C(d))$.

The following is an example of a choice correspondence over $X = \{a, b, c, d\}$ which does not consist of a ‘stable’ set S with the properties that (i) $C(S) = S$ and (ii) there is no non-empty set A such that $A \cap S = \emptyset$ and $A \subseteq C(A \cup S)$.¹⁴ These properties are indispensable since these sets play an important role in CY-stability of matchings. If a firm or worker is matched to such a set then it cannot be blocked.

Example 3. Let $X = \{a, b, c, d\}$. Let the choices be as shown Table 4. We note that the choice correspondence does not have a ‘stable’ IR set even though it satisfies SUB. For example, even though $\{a, b, d\}$ is IR, $C(\{a, b, d\} \cup \{c\}) = C(X) = \{a, d\}$. Therefore, property (ii) mentioned above is not satisfied for any IR set. Similarly, singleton sets are also not stable since $C(a, b) = \{a, b\}$ (implies neither $\{a\}$ nor $\{b\}$ are stable, $C(c, d) = d$ ($\{c\}$ is not stable), $C(a, d) = \{a, d\}$ ($\{d\}$ is not stable).

The above choice correspondence does not satisfy PI either since $C(X) = d \neq C(C(a, b, d) \cup C(c)) = C(\{b, d\} \cup \{c\}) = C(\{b, c, d\}) = \{c, d\}$. The choice function does not satisfy GA since,

$$C(\{b, d\} \cup \{c\}) = \{c, d\}, C(\{c, d\} \cup \{a\}) = \{a, d\}, C(\{a, d\} \cup \{b\}) = C(\{a, b, d\}) = \{b, d\}.$$

In other words, GA is violated if we take the following sets ($k = 4$) as per the definition,

$$S_1 = \{b, d\}, S_2 = \{c\}, D_1 = \{d\}, S_3 = \{a\}, D_2 = \{d\}, S_4 = \{b\}.$$

Therefore, this is a form of acyclicity in revealed preferred choices when alternatives are ‘removed’ and ‘added’. We show that GA along with SUB is sufficient for the existence of a CY-stable matchings. We now provide the main result.

Theorem 1. *Suppose the choice correspondences of firms and workers satisfy SUB and GA. Then there exists a CY-stable many-to-many matching.*

Theorem 1 provides weaker sufficient conditions on choice correspondences for the existence of CY-stable matchings.

The proof is provided in the Appendix. We provide a sketch of the proof. First, we show that there must exist a strongly maximal IR set as defined earlier using a

¹⁴We call these sets strongly maximal IR sets in the proof of the theorem.

Subset(s)	$C(\cdot)$
X	$\{d\}$
$\{a, b\}$	$\{a\}$
$\{a, c\}$	$\{a, c\}$
$\{a, d\}$	$\{a, d\}$
$\{b, c\}$	$\{b, c\}$
$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{c, d\}$
$\{a, b, c\}$	$\{a, c\}$
$\{a, b, d\}$	$\{b, d\}$
$\{a, c, d\}$	$\{a, d\}$
$\{b, c, d\}$	$\{c, d\}$
$\{x\}, x \in X$	$\{x\}$

Table 4: Choices which satisfies SUB but not GA

Grow or Discard Algorithm (GDA) for individual choice correspondences. We then use the Dynamic Grow or Discard Matching Algorithm (GDMA) to arrive at a stable matching through proposals and rejections. This algorithm is significantly different from DAA since rejected contracts are not discarded permanently by the proposing side. We prove the termination of the algorithm using finiteness of the set of all contracts and GA.

We present the GDA (Algorithm 1) in the table below which always produces a strongly maximal IR set.¹⁵ It starts with $C(\mathcal{X})$ and proceeds to add a disjoint set to it if indicated by the choice correspondence.

The algorithm is described from the perspective of an agent (firm or worker) which starts with the set $S = C(\mathcal{X})$ which is the set of contracts associated with the agent chosen from the grand set of contracts \mathcal{X} pertaining to that agent (i.e. \mathcal{X}_f for any firm $f \in \mathbf{F}$ or \mathcal{X}_w for any worker $w \in \mathbf{W}$). The next step is to look for a ‘challenger’ set i.e. a set which is disjoint from $C(\mathcal{X})$ and check if the agent would like to keep all the contracts from the challenger set. If this is not the case, then the algorithm proceeds to look for another ‘challenger’ set till it finds one. Otherwise the initial set is a strongly maximal IR set: IR holds due to SUB and strong maximality follows from the fact that there are no further contracts to add.

At any stage, if the algorithm finds a challenger set S' for the initial set S , it checks the following two conditions:

¹⁵See the proof for a formal definition.

Algorithm 1 Finding a Strongly Maximal IR Set

```
1: Let  $S_0 \leftarrow C(\mathcal{X})$  ▷ By SUB,  $S_0$  is an IR set. We start here
2: Let  $S \leftarrow S_0$ 
3: repeat
4:   Let  $found\_challenger \leftarrow false$ 
5:   Let  $\mathcal{D} \leftarrow \{S' \in \mathcal{X} \mid S' \neq \emptyset, S' \cap S = \emptyset\}$  ▷ The set of all disjoint challengers
6:   for all  $S' \in \mathcal{D}$  do
7:     ▷ Test for a “Growth” move or “Discard” move
8:     Let  $S_{union} \leftarrow S \cup S'$ 
9:     Let  $S_{choice} \leftarrow C(S_{union})$ 
10:    ▷ Check for a “Growth” move
11:    if  $S_{choice} = S_{union}$  then
12:       $S \leftarrow S_{union}$  ▷ Grow the set
13:       $found\_challenger \leftarrow true$ 
14:      break ▷ Restart the loop with the new, larger set S
15:      ▷ Check for a “Discard” move
16:      if  $S' \subseteq S_{choice}$  and  $S \not\subseteq S_{choice}$  then ▷ i.e.,  $S' \succ S$ 
17:         $S \leftarrow C_f(S_{choice})$  ▷ Move to the new set. This set is IR by SUB
18:         $found\_challenger \leftarrow true$ 
19:        break ▷ Restart the loop with the new set S
20: until  $found\_challenger = false$ 
21: return  $S$ 
```

(a) Does the agent ‘choose’ to add all the contracts in S' *in addition* to the set of contracts in $C(\mathcal{X})$, i.e., if $C(S \cup S') = S \cup S'$ or not? If the answer is yes, then the Algorithm moves to $S \cup S'$ and assigns this as the new S to find a challenger for this set.

(b) If part (a) does not hold for S' , the algorithm checks if the agent would like to add all contracts in S' by discarding some contracts that were present in $C(S)$. Formally, $C(S \cup S') \neq S \cup S'$ and $S' \subseteq C(S \cup S')$. If the latter holds then the algorithm moves to $C(S \cup S')$ and assigns this as the new set S and proceeds to look for a challenger to this set.

We now argue that the algorithm must stop in finite iterations: We use the finiteness of \mathcal{X}_f and the non-acyclicity of the “Discard” process to prove that it terminates. We define a binary relation \succ over \mathcal{X}_f as follows: $S \succsim S'$ if and only if $S \subseteq C(S \cup S')$, and $S \succ S'$ if and only if $S \succsim S'$ and $\neg(S' \succsim S)$. The “Growth” path (Line 11) moves to a strictly larger set. Since \mathcal{X}_f is finite, this path cannot continue forever. The “Discard” path: (Line 16) moves from S to a new set $S_{new} = C(S \cup S')$, where $S' \succ S$. This is a “move” along the \succ relation.

By GA, this “Discard” path cannot loop forever (as that would be a cycle of strict relations which is not allowed under GA). Since the algorithm can never be in an infinite

loop (it is always moving to a “new” state, either by growing or by an acyclic discard to a new set), and the number of states (i.e. $|\mathcal{X}|$) is finite, the algorithm is guaranteed to terminate.¹⁶

Why the final set is a strongly maximal IR set: The algorithm stops (at Line 21) only when it fails to find a ‘challenger’ set. This means it has found a set S such that:

- (i) It is IR (all sets S in the algorithm are IR sets).
- (ii) The “Growth” test (Line 11) failed for all challengers S' .
- (iii) The “Discard” test (Line 16) failed for all challengers S' .

The failure of the “Growth” test means there is no disjoint S' such that $C(S \cup S') = S \cup S'$ i.e. the firm f does not want to add any more contracts pertaining to her. This is equivalent to proving that there is no S' such that $S' \succsim S$ given that $S \succsim S'$. The failure of the “Discard” test means there is no S' such that $S' \succ S$. In other words, there are no contracts which can be added further and the remaining ones are revealed preferred to any other set S' . Therefore, the GDA ends with a strongly maximal IR set S defined as follows: set S is strongly maximal if (i) $C(S) = S$ and (ii) there is no non-empty set $S' \cap S = \emptyset$ such that $S' \subseteq C(S \cup S')$.

All the above arguments hold for the preferences of the workers as well. All the above arguments hold for the preferences of the workers as well. We now describe the Dynamic Grow or Discard Matching Algorithm (GDMA), which adapts the logic of CY into a path-dependent environment where rejected contracts are not permanently discarded.

Step 1 (Initialization): Let the initial tentative matching be $\mu^0 = \emptyset$. In the first stage, each firm f identifies a strongly maximal individually rational (IR) set from its universal set of contracts X_f (existence is guaranteed by Lemma 4) and proposes it. Let O^1 be the union of all these proposed contracts. Each worker w tentatively accepts $\mu_w^1 = C_w(O_w^1)$, where O_w^1 is the pool of all contracts naming w offered by all firms in the first stage. If a firm has multiple strongly maximal IR sets, a fixed tie-breaking rule is used. Note that μ_w^1 is inherently IR since, by SUB, $C_w(C_w(O_w^1)) = C_w(O_w^1)$. Contracts not chosen by the workers are rejected and returned to the firms, but they are *not* permanently discarded.

¹⁶There may be ‘mixed’ cycles where some steps in the path involved “growing” the set and some may involve discarding. These cycles can be converted to ‘discard’ cycles as follows: Suppose $A \rightarrow A \cup B \rightarrow C \rightarrow D \rightarrow A$ is one such cycle where $A \cap B = (A \cup B) \cap C = C \cap D = \emptyset$. Then this generates a discard cycle: $A \cup B \rightarrow C \rightarrow D \rightarrow A \cup B$. This can then be ruled out due to GA. See the proof of Theorem 2 in the Appendix for details.

Step 2 (Iterative Proposals and Transitions): At any stage $k > 1$, let the current tentative matching be μ^{k-1} . From the firm’s perspective, if a firm f ’s current matched set μ_f^{k-1} is not strongly maximal, it may attempt to ‘grow’ or ‘discard’. Specifically, if there exists a disjoint set of contracts $O_f^k \subseteq X_f \setminus \mu_f^{k-1}$ (which may freely include contracts rejected in previous stages) such that $O_f^k \subseteq C_f(\mu_f^{k-1} \cup O_f^k)$, firm f proposes O_f^k . Let O^k be the union of all such proposals made at stage k .

From the worker’s perspective, workers follow the logic of the GDA to evaluate these new proposals. If a worker w is currently matched to μ_w^{k-1} and receives a new pool of offers O_w^k , the worker evaluates the union of their current holdings and the new offers: $C_w(\mu_w^{k-1} \cup O_w^k)$.

- If $O_w^k \not\subseteq C_w(\mu_w^{k-1} \cup O_w^k)$, the new proposal is not a threat, and the worker rejects the disruptive contracts, maintaining their optimal choice.
- If $O_w^k \subseteq C_w(\mu_w^{k-1} \cup O_w^k)$, the proposal successfully challenges the current set. The worker accepts the transition, and their new tentative match becomes $\mu_w^k = C_w(\mu_w^{k-1} \cup O_w^k)$.

By doing so, the worker may discard some previously held contracts to accommodate the new ones. Because $C_w(C_w(\mu_w^{k-1} \cup O_w^k)) = C_w(\mu_w^{k-1} \cup O_w^k)$, the newly transitioned set remains IR.

Step 3 (Termination): The algorithm terminates at a stage T when no firm can make a valid proposal that the workers are willing to accept. That is, the matching transitions stop when there exists no firm f and non-empty proposal O_f such that $O_f \subseteq C_f(\mu_f^T \cup O_f)$ and $O_w \subseteq C_w(\mu_w^T \cup O_w)$ for all involved workers w .

Convergence of the Dynamic GDMA: Unlike standard deferred acceptance algorithms, firms in the Dynamic GDMA do not permanently discard rejected contracts. Consequently, a worker may be re-offered a contract they previously rejected, meaning the set of active contracts in the market does not shrink monotonically. However, GA strictly ensures that the sequence of choice transitions for any agent cannot cycle back to revive a previously discarded state. Under the GDMA, every successful proposal forces the involved agents to transition strictly along their revealed preference paths. Because the universal set of contracts X is finite, the state space of all possible matchings is also finite. Since GA strictly rules out cycles in the transition path, the algorithm travels through a finite path of matchings. Therefore, the process must converge in finite time, terminating at a state where no further valid blocking proposals can be made. We

formally prove the stability of this terminal matching in the Appendix.

4.1 On the existence of a lattice structure

The classical lattice structure of stable matchings relies on the assumption that market negotiations move in a single, irreversible direction. In standard models, once a contract is rejected, it is permanently discarded. This creates a monotonic progression that naturally converges to a universally “firm-optimal” or “worker-optimal” outcome. However, by allowing for context-dependent choices and relaxing the Irrelevance of Rejected Contracts, our model introduces inherent path-dependence into the negotiation process. A worker might reject a contract initially, only to find it desirable later if a new complementary contract changes the viability of their available menu. Because agents can dynamically adjust their choices and firms can re-propose previously discarded options, the trajectory of the market is no longer a fixed one-direction path, but a sequence of past adjustments. Therefore, the specific stable matching reached by the GDMA is fundamentally tied to the exact sequence of proposals and transitions.

Therefore, our conjecture is that the set of stable matchings in this generalized environment is unlikely to preserve the classical lattice structure. While our Dynamic GDMA guarantees that a stable matching will always be reached, the universal existence of a “firm-best” or “worker-best” matching suffers due to our behavioral generalization. This leaves the formal structural characterization of the stable set as a compelling open question for future research.

4.2 On the necessity of the conditions: Substitutability and General Acyclicity

A natural question arises regarding the strict necessity of Substitutability and General Acyclicity for the existence of stable matchings. In matching theory, existence is highly dependent on the specific profile of the broader market. It is entirely possible to construct isolated market instances where an agent violates both Substitutability and General Acyclicity exhibiting severe complementarities or cyclical choice behavior, yet a stable matching still happens to exist due to delicate alignments in the other agents’ choices. As a result, strict point-wise necessity cannot be claimed. However, in the matching with contracts literature, necessity is typically understood through the lens of a maximal domain (e.g., [Hatfield and Kojima \(2008\)](#)). In this broader sense, while a stable outcome might exist by chance in a specific market without these conditions, their absence destroys the guarantee of existence across all possible market configurations.

Within the context of our generalized framework, Substitutability and General Acyclicity function as the fundamental boundary conditions required for algorithmic convergence. Substitutability is necessary to prevent the market from collapsing under standard complementarities, ensuring that the withdrawal of an offer does not lead to a cascading of unrelated rejections. General Acyclicity, which replaces the overly restrictive Irrelevance of Rejected Contracts (IRC), is the minimal structural requirement necessary to prevent our Dynamic GDMA from falling into infinite loops of path-dependent choices. While our combined conditions are strictly weaker than the Path Independence required by [Chambers and Yenmez \(2017\)](#), we view them as the frontier for guaranteeing stability. Whether Substitutability and General Acyclicity strictly define the absolute maximal domain for stable matchings in non-IRC environments remains a delicate and important question for future research.

5 The one-to-one matching problem

Let $|\mathbf{F}| = |\mathbf{W}| = m$. In this section we restrict attention to one-to-one matchings, i.e., $|\nu_i| \leq 1$ for all $i \in \mathbf{F} \cup \mathbf{W}$. We define it formally.

Definition 7 (One-to-One Matching). A one-to-one matching is a matching $\nu \subseteq \mathbf{X}$ such that for every agent $i \in \mathbf{F} \cup \mathbf{W}$, $|\nu_i| \leq 1$. Therefore, in a one-to-one matching each agent is involved in at most one contract.

Note that this also implies that each firm is uniquely matched to a worker (and vice versa) through one contract. However, the set of contracts could involve multiple contracts between the same worker-firm pair. We prove the axiom used for our stability result.

5.1 Axioms

We provide an axiom which is used to prove the existence of stable one-to-one matchings.

Definition 8 (Binary acyclicity (BA)). A choice correspondence C satisfies BA if for an array or sequence of alternatives (contracts) $(x_1, x_2, x_3, \dots, x_k) \in \mathbf{X}^k$, $[x_1 \in C(x_1, x_2), x_2 \in C(x_2, x_3), \dots, x_{k-1} \in C(x_{k-1}, x_k)] \Rightarrow [x_1 \in C(x_1, x_k)]$.

BA requires that choices from pairwise sets be acyclic in the sense that if x_1 is chosen from the set $\{x_1, x_2\}$, x_2 is chosen from $\{x_2, x_3\}$ and so on till x_{k-1} is chosen from $\{x_{k-1}, x_k\}$ then x_1 must be chosen from $\{x_k, x_1\}$. For example, if $C(a, b) = \{a\}$, $C(b, c) = \{b\}$ and $C(a, c) = \{c\}$ violates BA and is problematic for rationalizability. However, BA is weaker

than that since $a = C(a, b)$, $b = C(b, c)$ and $C(a, c) = \{a, c\}$ satisfies BA. We now argue that PI implies BA.

Lemma 2. Suppose choice correspondence $C : \mathcal{X} \rightarrow \mathcal{X}$ satisfies PI. Then, C satisfies BA.

Proof. Suppose PI holds but BA does not hold. Then there exists a sequence of alternatives $(x_1, x_2, x_3, \dots, x_k) \in \mathbf{X}^k$ such that $x_1 \in C(x_1, x_2), x_2 \in C(x_2, x_3), \dots, x_i \in C(x_i, x_{i+1}), \dots, x_{k-1} \in C(x_{k-1}, x_k)$ and $x_1 \notin C(x_1, x_k)$. Since $\text{PI} \implies [\text{SUB} + \text{CON}]$, we use the latter two properties. By contrapositive of CC, $x_1 \notin C(x_1, x_k)$ implies $x_1 \notin C(x_1, x_2, \dots, x_k)$. By CON, $[x_1 \notin C(x_1, x_2, \dots, x_k)] \implies [x_1 \notin C(x_1, x_2)]$. This is a contradiction. ■

CY-stability is not ideal for this setting due to the set inclusion property. We define a new notion of stability based on replacement of a current match.

Definition 9 (R-Stability). A matching $\nu \in \mathcal{X}$ is R-stable if it satisfies the following two properties:

1. Individual rationality (IR): A matching is said to be IR if $C_i(\nu_i) = \nu_i$ for all $i \in \mathbf{F} \cup \mathbf{W}$.
2. No blocking: There exists no contract $x \notin \nu$ such that $C_i(\nu_i \cup \{x\}) = \{x\}$ for $i \in \{f(x), w(x)\}$.

Therefore, stability in the one-to-one matching problem requires two properties: (i) the choice from the matched set should be the set itself and (ii) no other firm-worker pair should deviate by ‘replacing’ their current contract with another contract involving them.

The notion of blocking requires ‘replacement’ rather than requiring set inclusion of blocking contracts and is similar to the one used in [Hatfield and Milgrom \(2005\)](#) and is more relevant in this section because in the one-to-one matching problem every profitable deviation necessarily takes the form of complete replacement. Therefore, requiring equality captures the more economically meaningful deviation in this setting.

We provide an example to show that PI is not necessary for the existence of R-stable one-to-one matchings. In this simplified example, the set of worker-pair contracts are just the set of all possible contracts.

Example 4. Let $\mathbf{F} = \{f_1, f_2, f_3\}$ and $\mathbf{W} = \{w_1, w_2, w_3\}$. In this example we assume that $\mathbf{X}_f = \{w_1, w_2, w_3\}$ for each $f \in \mathbf{F}$ and $\mathbf{X}_w = \{f_1, f_2, f_3\}$ for each $w \in \mathbf{W}$. The preferences of firms and workers are given in the tables below.

	Firms		
Contracts	f_1	f_2	f_3
$\{w_1, w_2\}$	$\{w_2\}$	$\{w_1\}$	$\{w_1\}$
$\{w_1, w_3\}$	$\{w_1, w_3\}$	$\{w_1, w_3\}$	$\{w_1\}$
$\{w_2, w_3\}$	$\{w_3\}$	$\{w_2\}$	$\{w_2, w_3\}$
$\{w_1, w_2, w_3\}$	$\{w_1\}$	$\{w_1, w_2, w_3\}$	$\{w_3\}$

(a) Firm Preferences

	Workers		
Contracts	w_1	w_2	w_3
$\{f_1, f_2\}$	$\{f_2\}$	$\{f_1\}$	$\{f_2\}$
$\{f_1, f_3\}$	$\{f_1, f_3\}$	$\{f_1\}$	$\{f_1\}$
$\{f_2, f_3\}$	$\{f_3\}$	$\{f_2, f_3\}$	$\{f_2, f_3\}$
$\{f_1, f_2, f_3\}$	$\{f_1, f_2, f_3\}$	$\{f_1, f_2, f_3\}$	$\{f_1, f_2\}$

(b) Worker Preferences

Table 5: Choice correspondences for Example 4

All the agents' choice correspondences violate PI. However, the matching $\nu(w_1) = f_3, \nu(w_2) = f_1, \nu(w_3) = f_2$ is R-stable. Hence, PI is not necessary for stability.

Our first result shows that binary acyclicity (BA) is sufficient for the existence of R-stable matching in the one-to-one matching problem. We have already shown in Lemma 2 that PI implies BA.

Theorem 2. *Suppose choice correspondences of firms and workers satisfy BA. Then there exists a R-stable one-to-one matching.*

The proof of the theorem is provided in the Appendix. Here we provide an intuitive sketch of the proof. We first construct binary orders for all firms and workers using choices over binary sets of contracts. Given this construction, a firm proposing (worker proposing) Deferred Acceptance Algorithm (DAA) algorithm generates a matching, ν , where any firm (worker) not belonging to any choice set from a singleton set of contracts is never matched. If some 'revealed' preferences are not strict, we arbitrarily break ties as in Roth (2008) to run the DAA.

This matching, ν , is IR by construction, as each firm only proposes to a contract which is individually rational for the firm. If a distinct contract $x \notin \nu$ blocks the match, in the firm proposing DAA it must be that $C_f(\nu_f, x) = \{x\}$ and $C_w(\nu_w, x) = \{x\}$. However, this is not possible since f must have proposed to the more preferred contract first since both contracts are IR, and have been rejected in favor of some other contract x'' (or firm f''), which in turn gets rejected in favor of another contract x''' (or f'''), and this sequence eventually stops when some firm gets rejected in favor of a contract x' (or firm f'). By BA, $x' \in C_w(\{x', \nu_w\})$ is a contradiction. Therefore, the matching ν must be R-stable.

Therefore, the significance of Theorem 1 lies in showing that existence of R-stable one-to-one matchings does not require the full strength of PI. By using a weaker condition than PI, our theorem generalizes the results of the literature. In particular, it shows that the DAA can be applied using only binary choice information, without assuming PI. This also indicates that existence depends on local and pairwise choices, and that strong revealed-preference structure is not necessary for stability in one-to-one environments. We extend similar results to the many-to-many matching problem with the same notion of stability as the one used in the literature.

The replacement-based blocking notion used in the one-to-one environment reflects the capacity constraint that agents can hold at most one contract. In such environments, any profitable deviation necessarily involves replacing the currently held contract rather than adding an additional one. Requiring blocking to occur through replacement therefore aligns the stability concept with the feasible deviations available to agents and avoids considering deviations that are not behaviorally meaningful in one-to-one settings.

5.2 Discussion and Interpretation

This section discusses the behavioral interpretation of the conditions introduced in the paper and their role in guaranteeing the existence of stable outcomes.

The acyclicity conditions identified in our paper characterize a common condition in both models (one-to-one and many-to-many) on the choice correspondences which are required to guarantee existence of stable matchings. Choice cycles can generate situations in which agents alternately accept and reject contracts depending on the available menu, preventing the DAA-type algorithms from reaching a stable outcome. The conditions introduced in this paper restrict these cycles while allowing violations of stronger consistency properties such as Path Independence. As a result, they identify a minimal structure on choice behavior that is sufficient to guarantee stability. The matching algorithm used for the many-to-many model is significantly different from the DAA since rejected contracts cannot be discarded right away. The GDMA introduced in the paper relies on acyclicity in revealed choices to converge to a CY-stable matching.

The results highlight the distinct roles played by substitutability and acyclicity in matching models with choice correspondences. Substitutability ensures that the acceptability of a contract does not increase when other contracts are removed, preventing complementarities that can create issues for stability. Acyclicity, on the other hand, prevents chains of mutually reinforcing rejections that arise from inconsistent choice patterns. These properties ensure that stable outcomes exist even when agents' choices satisfy these weaker

properties.

Finally, the one-to-one and many-to-many results illustrate a common theme. In both environments, failures of stability arise primarily from cycles in revealed choices rather than from violations of substitutability alone (as in the many-to-many case). Binary acyclicity eliminates such cycles in pairwise comparisons, while general acyclicity extends this idea to sets of contracts. This parallel structure suggests that acyclicity plays a fundamental role in guaranteeing stable outcomes when agents are modeled using choice correspondences.

6 Conclusion

This paper studies the existence of stable matchings when agents have choice correspondences rather than preference relations. For the one-to-one matching problem, we show that Path Independence is not sufficient to guarantee R-stability. We show that if choice correspondences satisfy Binary Acyclicity, then R-stable one-to-one matchings exist.

For the many-to-many matching problem, we weaken the conditions required for the existence of CY-stable matchings relative to the existing literature. In particular, we show that Substitutability together with a new condition, General Acyclicity, is sufficient to guarantee the existence of CY-stable matchings. Our proof provides a constructive procedure—the Dynamic Grow or Discard Algorithm—which iteratively expands or revises sets of contracts until a strongly maximal individually rational set is reached—resulting in a CY-stable matching. This result shows that path independence is not necessary for stability: the combination of Substitutability and General Acyclicity is strictly weaker while still ensuring existence.

More broadly, this paper adds to the growing literature that examines matching markets through observable choice behavior. By identifying the basic behavioral conditions that ensure stability, this analysis clarifies the foundations of matching with non-standard choice behavior. It also provides new insights in matching market settings where preference information is incomplete or difficult to elicit.

For future research, it may be fruitful to explore whether additional structural properties of stable matchings (e.g. lattice structures) or results from comparative statics can be achieved under weaker conditions. Another approach is to investigate the incentive and strategic properties of mechanisms when agents are described by choice correspondences. Finally, applying these ideas to dynamic or large matching markets may also be useful in these contexts for boundedly rational or context-dependent choice behavior.

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Appendix

Proof of Theorem 2

Proof. We will construct binary relations for both sides of the market. Consider a firm $f \in \mathcal{F}$, we define the binary relation \succsim_f over $\mathcal{X}_f \cup \emptyset$, where we denote the asymmetric and symmetric parts by \succ_f and \sim_f respectively, as follows:

1. $x \succsim_f x$ for all $x \in \mathcal{X}_f$, and $\emptyset \succsim_f \emptyset$.
2. (a) $x \succsim_f \emptyset$ iff $x \in c_f(\{x\})$.
 (b) $\emptyset \succsim_f x$ iff $x \not\prec_f \emptyset$.
3. (a) If $x \succsim_f \emptyset$ and $\emptyset \succsim_f x'$, then $x \succ_f x'$.
 (b) If $\emptyset \succsim_f x$ and $\emptyset \succsim_f x'$, then $x \succsim_f x'$ and $x' \succsim_f x$.
4. If 3(a) and 3(b) do not apply:
 - (a) If $x = c_f(\{x, x'\})$, then $x \succsim_f x'$ and $x' \not\prec_f x$,
 - (b) If $x \succ_f x'^{17}$ and $x' \succsim_f x''$, then $x \succ_f x''$,
 - (c) If $x \succsim_f x'$, $x' \succ_f x''$, then $x \succ_f x''$,
 - (d) If 4(a), 4(b) and 4(c) do not apply for $x, x' \in \mathcal{X}$, then $x \succsim_f x'$ and $x' \succsim_f x$.

Lemma 3. \succsim_f is an complete, reflexive and transitive.

Proof. (i) By (1), \succsim_f is reflexive by construction.

(ii) We now show completeness:

By 2(a) and 2(b) the relation is always defined between the empty set and any $x \in \mathcal{X}_f$. Between any pair of x, x' , if 3(a) holds, then the relation is defined.

¹⁷Recall that $x \succsim_f x'$ and $x' \not\prec_f x \Rightarrow x \succ_f x'$.

If 3(a) does not hold, then because either $c_f(\{x, x'\})$ is empty or non-empty, by 3(b) and 4(a-d) the relation is defined.

(ii) To show transitivity, we look at the following cases:

Case 1: $\emptyset \succsim_f x$ and $x \succsim_f x'$ implies $\emptyset \succsim_f x'$.

In this case since $\emptyset \succsim_f x$, the only way $x \succsim_f x'$ is if 3(a) does not hold, i.e. $x' \not\prec_f \emptyset$. Then 2(b) gives us $\emptyset \succsim_f x'$ as required.

Case 2: $x \succsim_f \emptyset$ and $\emptyset \succsim_f x'$ implies $x \succsim_f x'$.

This is true, straightforwardly from 3(a).

Case 3: $x \succsim_f x'$ and $x' \succsim_f \emptyset$ implies $x \succsim_f \emptyset$.

If $x \succsim_f \emptyset$ is not true, then by 2(b) we have $\emptyset \succsim_f x$, but then $x \succsim_f x'$ violates 3(a).

Case 4: $x \succsim_f x'$ and $x' \succsim_f x''$ implies $x \succsim_f x''$.

- if both $\emptyset \succsim_f x$ and $\emptyset \succsim_f x''$, then by 3(b), $x \succsim_f x''$.
- if $x \succsim_f \emptyset$ and $\emptyset \succsim_f x''$ then by 3(a), $x \succsim_f x''$.
- it cannot be that $x'' \succsim_f \emptyset$ and $\emptyset \succsim_f x$ because then $x' \succsim_f x''$ implies by Case 2(b) that $\emptyset \succsim_f x'$ which coupled with $x' \succsim_f x''$ violates 3(a).
- if both $x \succsim_f \emptyset$ and $x'' \succsim_f \emptyset$, then by 3(a) we must have $x' \succsim_f \emptyset$.
 - if $x = c_f(\{x, x'\})$ and $x' = c_f(\{x', x''\})$, by BA and 4(a) we have $x \succsim_f x''$.
 - we can have $x \not\prec_f x''$ if $c_f(\{x, x''\}) = \{x''\}$, but then 4(b) implies $x'' \succsim_f x'$ and $x' \not\prec_f x''$ which is a contradiction.
 - the only other way we can have $x \not\prec_f x''$ is if there exists a sequence of $\{x_1, \dots, x_n\}$ such that $x_i \succsim_f x_{i+1}$ for all $i \in \{1, \dots, n-1\}$, $x_1 = x''$, $x_n = x$ and $x_i = c_f(\{x_i, x_{i+1}\})$ for some $i \in \{1, \dots, n-1\}$. Since, $x \succsim_f x'$, the only way $x' \succsim_f x''$ can be ensured is if 4(a) holds, i.e. $c_f(\{x', x''\}) = x'$, but then by 4(c) we have that $x \succsim_f x''$ which is a contradiction.

■

So, this binary relation is an order. We construct binary orders analogously for all firms and workers $((\succsim_f)_{f \in \mathcal{F}}, (\succsim_w)_{w \in \mathcal{W}})$. Given this construction, a firm proposing (worker proposing) DAA generates a matching where no firm (worker) that offers a feasible contract which is not individually rational for the worker (firm) it offers it to get matched to that worker (firm) through that contract. If some preferences are not strict, we arbitrarily

break ties (Roth (2008)) to run the DAA.

This matching is individually rational by construction, as each firm only proposes contracts to a worker which is individually rational for the firm. Consider a blocking contract x which blocks the matching in the sense of stability, in the firm proposing DAA. Denote by (f, w) the firm and worker pair associated with x . It must be that $C_f(\{\nu_f, x\}) = x$ and $C_w(\{\nu_w, x\}) = x$ where ν_f is ν_w 's are the contracts matched to the worker and the firm in the firm proposing DAA. But then f must have proposed the contract x first since it is individually rational as it is preferred to ν_f and been rejected in favor of a preferred match x' which is rejected by w in favor of x' , which in turn gets rejected in favor of x'' and this sequence eventually stops when the worker accepts ν_w and does not reject it. By BA, we then have $x \in c_w(\{x', \nu_w\})$, which is a contradiction. Thus, such a matching will be stable. ■

Proof of Theorem 1

Proof. We will construct a binary relation which will be used to construct a stable matching. We use the GDA (Algorithm 1), which compare pairs of disjoint sets of contracts in \mathcal{X}_f (or \mathcal{X}_w). This is due to the fact that if a firm f is matched with a set of contracts $X \in \mathcal{X}_f$, then it can only be blocked with another non-empty disjoint set of contracts $X' \in \mathcal{X}_f$. We introduce some notation to specify the binary relation over disjoint sets of contracts.

Let $\overline{\mathcal{X}}_f$ denote the set of all non-empty disjoint pairs of sets in \mathcal{X}_f , i.e.,

$$[(X', X'') \in \overline{\mathcal{X}}_f] \text{ if and only if } [X', X'' \in \mathcal{X}_f \setminus \emptyset \text{ and } X' \cap X'' = \emptyset].$$

For each firm $f \in \mathcal{F}$, we can define the binary relation, \succsim_f , on $\overline{\mathcal{X}}_f$ as follows: for any $(X', X'') \in \overline{\mathcal{X}}_f$,

$$[X' \succsim_f X''] \iff [X' \subseteq C_f(X' \cup X'')].$$

In other words, a set of contracts is weakly better if it is weakly ‘revealed preferred’ to the other as given by the choice correspondence. We say that $X \succ_f X'$, if and only if $X \succsim_f X'$ and $\neg(X' \succsim_f X)$. Moreover, $X \sim_f X'$ if and only if $X \succsim_f X'$ and $X' \succsim_f X$. Note that if $X \succsim_f X'$ then $C_f(X \cup X') = X \cup X'$. Note that \succsim_f is not reflexive over $\overline{\mathcal{X}}_f$ since $(X', X') \notin \overline{\mathcal{X}}_f$ for any $X' \in \mathcal{X}_f$. It is also not complete over $\overline{\mathcal{X}}_f$ since it is possible

to have $\neg(X' \succsim_f X'')$ and $\neg(X'' \succsim_f X')$ for some $(X', X'') \in \overline{\mathcal{X}}_f$. In such cases, we say that X' and X'' are *non-comparable* according to \succsim_f and write $X' \parallel_f X''$. Note that for stability purposes, non-comparability between two subsets is not an issue. Suppose a firm is matched with a set of contracts $X \in \mathcal{X}_f$ and another set of contracts $X' \in \mathcal{X}_f$ is offered such that $X \cap X' = \emptyset$. Since $X \parallel_f X'$, we have $X' \not\subseteq C_f(X \cup X'')$ which implies that X' cannot block X (from the perspective of the firm). In the next lemma show that \succsim_f is acyclic in the strict component \succ_f .

We define **maximality** of a set with respect to the weak relation \succsim_f for the matching algorithm as follows.

Definition 10 (Maximality). For any set $X \in \mathcal{X}_f$,

$$[X \text{ is maximal w.r.t. } \succsim_f \text{ over } \overline{\mathcal{X}}_f] \iff [\nexists X' \in \mathcal{X}_f, X \cap X' = \emptyset \text{ s.t. } X' \succ_f X].$$

In the rest of the paper, we refer to the maximality of a set with respect to the weak relation \succsim_f . Note that for the grand set \mathcal{X}_f there is no non-empty disjoint set $X' \in \mathcal{X}_f$ such that $\mathcal{X}_f \cap X' = \emptyset$. Therefore, \mathcal{X}_f is trivially maximal. However, there may be many non-empty maximal sets.¹⁸

Maximality is defined in the weak sense which is not sufficient for the CY-stability of a matching since firms (or workers) may want to add more contracts. We say that a set X^* is strongly maximal if there does not exist any other non-empty set X' where $X^* \cap X' = \emptyset$, such that $X' \succsim_f X^*$. We say that a choice function is *trivial* if $C(S) = \emptyset$ for all $S \in \mathcal{X}_f$. Note that in this case only the empty set is strongly maximal. We say that $C(\cdot)$ is *non-trivial* if it is not trivial.

Lemma 4. There exists at least one strongly maximal set which is IR.

Proof. Suppose $C_f : \mathcal{X}_f \rightarrow \mathcal{X}_f$ is trivial. Then it is easy to check that \emptyset is the only strongly maximal IR set since for any other $X \in \mathcal{X}_f \setminus \emptyset$, $X \not\subseteq C_f(\emptyset \cup X)$. Therefore, $\emptyset \succ_f X$ for all $X \in \mathcal{X}_f$.

Suppose $C_f : \mathcal{X}_f \rightarrow \mathcal{X}_f$ is non-trivial. Algorithm 1 proves the existence of a strongly maximal IR set in this case. We provide a description of the algorithm. If C_f is non-trivial, then we can start with $C_f(\mathcal{X}_f)$ i.e. the choice from the grand set. By SUB, $C_f(C_f(\mathcal{X}_f)) = C_f(\mathcal{X}_f)$ which is IR.¹⁹ The Grow or Discard Algorithm (GDA) (Algorithm

¹⁸Note that the \emptyset is maximal since $\emptyset \subseteq C_f(X' \cup \emptyset)$ for all $X' \in \mathcal{X}_f$.

¹⁹Note that empty sets are also IR.

1) looks for a “challenger” set which is a disjoint set $X \in \mathcal{X}_f$ such that $X \cap C_f(\mathcal{X}_f) = \emptyset$ and $X \succsim_f C_f(\mathcal{X}_f)$. If $X \sim_f C_f(\mathcal{X}_f)$ then we move to $C_f(X \cup C_f(\mathcal{X}_f))$ by “adding” the disjoint set X . If $X \succ_f C_f(\mathcal{X}_f)$ then we move to $C_f(X \cup C_f(\mathcal{X}_f))$.²⁰ Note that in both these steps we move from an IR set to an IR set. This way, at each step, the resulting set is IR. At each point t we check if there is a challenger set X_{t+1} which can challenge the set at X_t : (i) if $X_{t+1} \sim_f X_t$ then by moving to $C_f(X_t \cup X_{t+1})$ we “grow” to this new set which consists of all the elements in $C_f(X_t)$ and (ii) if $X_{t+1} \succ X_t$ by moving to $C_f(X_t \cup X_{t+1})$ we “discard” $C_f(X_t) \setminus C_f(X_t \cup X_{t+1})$. If $X_{t+1} \parallel_f X_t$, the set X_{t+1} is not a threat to X_t and the algorithm then looks for a new challenger.

The Algorithm 1 ensures that the process must stop at some point if we start from $C(\mathcal{X}_f)$: (i) since there are no strict cycles with \succ_f and (ii) since \mathcal{X}_f is finite. Therefore, it must end with a *strongly maximal IR* set $X^* \in \mathcal{X}_f$ at which point there is no other X such that $X \succ_f X^*$ or $X \sim_f X^*$.²¹ We prove this formally below. ■

Lemma 5. The GDA (Algorithm 1) terminates at some finite stage $K \in \mathbb{Z}_{++}$ with a strongly maximal IR set $X^* = X_K$.

Proof. The Grow or Discard Algorithm (GDA) (Algorithm 1) that starts with the set, $C_f(\mathcal{X}_f)$, and “grows” by adding some elements without discarding any elements or “moves” to a new set by adding a new set and by discarding some elements until it finds one that is also strongly maximal. Note that if C_f satisfies SUB and $S = C_f(\mathcal{X}_f)$, then $C_f(S) = S$. Thus the chosen contracts are individually rational at each stage.

We now argue why the algorithm must stop, and why it must end up with a strongly maximal IR set. To prove this formally, note that it is sufficient to prove that no two sets reached at any two stages in the algorithm will be the same, i.e., $X_t \neq X_{t'}$ for any $t, t' \in \mathbb{Z}_+$ where $t \neq t'$. Since \mathcal{X}_f is finite, there are finite subsets of \mathcal{X}_f . Therefore, if sets at each stage are unique, there cannot be any cycles and the algorithm is guaranteed to stop at some stage K . Moreover, since each agent’s choice function satisfies SUB, any contract chosen from a feasible set remains chosen when the set contracts to the final assignment, implying IR.

Let X_t for $t \in \mathbb{Z}_+$ denote the IR set at time t which is reached by the algorithm. Suppose for contradiction that $X_t = X_{t'}$ for some $t, t' \in \{1, 2, \dots, K\}$ with $t \neq t'$. The following

²⁰The discarding will not happen at the first stage due to SUB, $x \in C_f(\mathcal{X}_f) \implies x \in C_f(X)$ for all $X \in \mathcal{X}_f$. However, it can happen at other stages in the algorithm.

²¹Recall that every set $C_f(X_t \cup X_{t+1})$ at any stage t where X_{t+1} is a challenger chosen in the Algorithm is an IR set.

sequence of steps represent this: $X_t \rightarrow \dots \rightarrow X_{t'}$. This implies the existence of distinct sets X_1, X_2, \dots, X_k such that,

$$\begin{aligned}
C_f(X_1) &= X_1 = X_t, \\
C_f(X_1 \cup X_2) &= X_2 \cup D_1 = X_{t+1}, D_1 \subseteq X_1, \\
C_f((X_2 \cup D_1) \cup X_3) &= X_3 \cup D_2, D_2 \subseteq X_2 \cup D_1, \\
&\vdots \\
C_f((X_{k-1} \cup D_{k-2}) \cup X_k) &= X_k \cup D_{k-1} = X_{t'}, D_{k-1} \subseteq X_{k-1} \cup D_{k-2},
\end{aligned}$$

where $X_1 = X_t$, $X_k \cup D_{k-1} = X_{t'}$ and $(X_{i-1} \cup D_{i-2}) \cap X_i = \emptyset$ for each $i \in \{3, \dots, k\}$.²² Note that for any two sets, X_t and X_{t+1} if $X_t \subseteq X_{t+1}$, we can consider the sequence with X_{t+1} in place of both X_t and X_{t+1} . Therefore, we can only consider the ‘discard’ path since the ‘grow’ path can be combined as follows. We relabel all such sets as X_t^* such that (i) $X_t^* = X_{t'}$ for some $t, t' \in \mathbb{Z}_+$, (ii) for any $X_t^* = X_{t'}, X_{t''}^* = X_{t'''}, [t < t''] \iff [t' < t''']$ and (iii) $X_t^* \not\subseteq X_{t+1}^*$ for all $t \in \mathbb{Z}_+$. Therefore, we can now rewrite the previous sequence of sets as follows:

$$\begin{aligned}
C_f(X_1^*) &= X_1^* = X_{t'}, \\
C_f(X_1^* \cup X_2^*) &= X_2^* \cup D_1^* = X_{t'+1}^*, D_1^* \subsetneq X_1^*, \\
C_f((X_2^* \cup D_1^*) \cup X_3^*) &= X_3^* \cup D_2^*, D_2^* \subsetneq X_2^* \cup D_1^*, \\
&\vdots \\
C_f((X_{k-1}^* \cup D_{k-2}^*) \cup X_k^*) &= X_k^* \cup D_{k-1}^* = X_{t'+k}^*, D_{k-1}^* \subsetneq X_{k-1}^* \cup D_{k-2}^*,
\end{aligned}$$

where $X_k^* = X_{t'+k}$, $D_{k-1}^* = \emptyset$ and $(X_{i-1}^* \cup D_{i-2}^*) \cap X_i^* = \emptyset$ for each $i \in \{3, \dots, k\}$. However, this is ruled out by GA. Therefore, at any two distinct stages in the algorithm, the two sets must be distinct. Since $|\mathcal{X}_f|$ is finite, the algorithm must end at some finite $K \in \mathbb{Z}_+$. Therefore, by construction, the algorithm provides a set $X^* = X_K$ which is IR and strongly maximal. \blacksquare

Similar properties can be shown for $C_w(\cdot)$ for every $w \in \mathbf{W}$. The firm proposing Dynamic Grow or Discard Matching Algorithm (GDMA) was presented on page 18. We show that the matching algorithm will produce a stable matching. Let ν be the final matching produced when the GDMA terminates. To prove that ν is a stable matching, we must show that it is individually rational (IR) and that there exists no blocking coalition.

²²The sets $\{D_i\}_{i=1}^{k-1}$ can be defined as the residual sets $D_i = C_f(X_i \cup X_{i+1}) \setminus X_{i+1}$ for $i \in \{1, \dots, k-1\}$.

Step 1: Termination

Unlike standard deferred acceptance algorithms, the Dynamic GDMA allows firms to re-propose previously rejected contracts. However, the algorithm strictly requires that any transition to a new tentative matching must be strictly validating for both the proposing firm(s) and the accepting workers. Specifically, a worker w only transitions to a new state if the new offer O_w acts as a valid challenger: $O_w \subseteq C_w(\nu_w^{k-1} \cup O_w)$.

Because the universal set of contracts X is finite, the state space of all possible matchings is finite. GA ensures that workers do not cycle through a sequence of chosen sets (i.e., the specific contradiction to GA detailed in the model rules out returning to a previously discarded state). Since the global matching state is driven by these non-cycling worker transitions, the algorithm moves through an acyclic path. Therefore, the GDMA must terminate in finite time.

Furthermore, by the construction of the algorithm, at every stage (including the final stage), each agent $i \in F \cup W$ applies their choice correspondence C_i to evaluate their holdings. Thus, $\nu_i = C_i(\nu_i)$ for all agents, ensuring the terminal matching ν is IR.

Step 2: Absence of a Blocking Coalition

Assume, for the sake of contradiction, that the terminal matching ν is not stable. Since ν is individually rational, there must exist a non-empty blocking coalition $Z \subseteq X$ such that $Z \cap \nu = \emptyset$, and for the relevant agents:

1. $Z_f \subseteq C_f(\nu_f \cup Z_f)$ for all firms f involved in Z , and
2. $Z_w \subseteq C_w(\nu_w \cup Z_w)$ for all workers w involved in Z .

In the Dynamic GDMA, firms are not permanently restricted to offer a rejected set of contracts. At any stage, a firm is permitted to propose any disjoint set of contracts $O_f \subseteq X_f \setminus \nu_f$ (including previously rejected ones) if it satisfies $O_f \subseteq C_f(\nu_f \cup O_f)$.

Suppose the hypothetical blocking coalition Z exists at the terminal matching ν . By the blocking condition, $Z_f \subseteq C_f(\nu_f \cup Z_f)$ holds for the firms. Therefore, according to the rules of the GDMA, the firms involved would actively propose the contracts in Z as a new offer ($O^T = Z$).

Furthermore, by the blocking condition for the workers, $Z_w \subseteq C_w(\nu_w \cup Z_w)$. According to the worker's transition rule in the GDMA, this means the proposal Z successfully challenges the workers' current holdings ν_w . The workers would therefore accept the proposal, resulting in a valid transition to a new matching state.

However, this directly contradicts the premise that the algorithm terminated at ν . By definition, the GDMA only reaches its terminal state when no such valid proposal Z can be made by the firms and accepted by the workers.

Therefore, the assumption that a blocking coalition Z exists must be false. Since ν is individually rational and admits no blocking coalitions, the matching ν produced by the GDMA is stable. ■