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Double auctions with complementary objects

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Abstract

Consider a situation where two objects that are complements are misallocated to different bidders. As a result, bidders try to trade via a double auction. The trade may be inefficient if the identities of buyer and seller are determined before they bid. To overcome inefficiency, we consider that the identities are determined by their bids. Under symmetric beliefs, we show that the bidders' expected buy and sell prices are equivalent for a rich family of bargaining rules. Consequently, bidders' expected utilities are equivalent under all bargaining rules. We also capture the impact of endogenous selection on the expected prices and utilities.

JEL classification: D44, D82
Keywords: bilateral trade, first-price double auction, second-price double auction, price-equivalence principle, payoff-equivalence principle

1 Introduction

If objects are complements, it leads to synergistic effects. Often, complementary objects are misallocated to different bidders which creates inefficiency. The misallocation occurs due to complex auction rules, uncertain future values, coordination failures, holdout problems, regulation, asymmetric information, liquidity constraints, etc. Certain examples are: airport slots, spectrum rights, mining rights, air rights, land parcels, patent portfolios, electricity grids, broadcasting rights, and so on.

An airline can work more efficiently if it can acquire adjacent departure and landing slots. To exploit economies of scale, a telecommunication firm wants to acquire spectrum rights for adjacent geographical regions. Two adjacent land parcels are worth more than two far-located land parcels. To develop a new technology, a firm has to acquire patents of other firms.

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In order to rectify the misallocation of complementary objects, agents attempt to trade their object with other agents via a double auction. However, trade may fail due to the following reasons: (a) both the agents want to be sellers, (b) both the agents want to be buyers, and (c) if one agrees to be a seller and other a buyer, they may not reach an agreeable price. As a result, the three scenarios lead to an inefficient outcome. In particular, (a) and (b) lead to trade failure with certainty while (c) leads to trade failure with a positive probability.

To overcome the trade failure, we devise an allocation rule that is efficient: buyer and seller are chosen endogenously, i.e., the identities of the agents are determined by their bids. To do a comparative analysis, we also consider the standard rule: buyer and seller are chosen exogenously, i.e., the identities of the agents are determined before they bid.

An obvious concern that arises is: does the endogenous selection criterion perform better than the exogenous selection criterion (part (c)) and the no-trade situation (parts (a) and (b))? The answer is yes, at least, for a special belief system.

Broadly, we address the following questions:

1. Given the endogenous selection of buyer and seller, what is the impact of bargaining power on the bid behavior and expected payoffs?
2. Does the expected buy price equal the expected sell price?
3. What is the impact of endogenous selection of buyer and seller on the bid behavior and expected prices at which trade occurs?

Consider a double auction with two bidders who possess one object each. The two objects are complements to each other and the value of each object is private information for the bidders. The degree of complementarity of the two objects is homogeneous and common knowledge among the bidders. The value spaces are subsets on the real line and the probability distributions are mutually independent.

Consider two environments: (a) *buyer-seller uncertainty* where the identities of the buyer and seller are endogenous and (b) *buyer-seller certainty* where the identities of the buyer and seller are exogenous. Under (a), the identities are determined by the bids. In particular, the highest bidder acts as a buyer while the lowest bidder acts as a seller. This ensures that trade occurs with certainty. Under (b), the identities are common knowledge before the agents bid and trade occurs if buyer's bid is at least as large as seller's bid.

In each of the two environments, we consider a family of bargaining rules where the market power is shared between the two bidders. In this family, two bargaining rules deserve special emphasis: (a) first-price double auction and (b) second-price double auction. Under the first-price double auction, the price of the object equals the buyer's bid, i.e., the buyer has all the market power. Under the second-price double auction, the price of the object equals the seller's bid, i.e., the seller has all the market power.[1]

Under buyer-seller uncertainty, we characterize an equilibrium with asymmetric beliefs for a generalized family of bargaining rules. With symmetric beliefs, we show that the bidders bid higher in the first-price double auction than the second-price double auction. We also establish two principles. The first is the price-equivalence principle which states that the bidders' *ex-ante* expected sell price equalizes their *ex-ante* expected buy price. The second is the payoff-equivalence principle which states that the bidders' *ex-ante* expected utilities are equivalent under all bargaining rules, i.e, the allocated market power is irrelevant in determining their expected utilities.

Under buyer-seller certainty, we characterize an equilibrium with asymmetric beliefs for a generalized family of bargaining rules. In the first- and second-price double auction, the inverse bid functions are derived analytically for a generalized family of probability distributions. We show that, under both the rules, the seller bids higher than the buyer as long as the degree of complementarity between the objects is not too large. In addition, if the seller is more likely to draw a higher value than the buyer, the buyer's belief about success is smaller than the seller's belief about failure. We also show that both the seller and buyer bid higher under the second-price double auction than the first-price double auction.

The rest of the results that are discussed below hold for a special belief system and the first-price double auction. In the buyer-seller uncertainty with asymmetric beliefs, we show that the price-equivalence principle fails. Particularly, if a bidder is more likely to draw a higher value, then his *ex-ante* expected buy price is larger than his *ex-ante* expected sell price. As a result, he induces higher *ex-ante* expected probability of being a buyer.

In the buyer-seller certainty with asymmetric beliefs, we show that the price-equivalence principle holds. We also show that the expected beliefs about trade succeeding are equivalent for buyer and seller. These properties hold without imposing stochastic orders on the belief system.

Our ultimate goal is to compare the bid behavior, expected prices and expected utilities between the buyer-seller uncertainty and buyer-seller certainty. With asymmetric beliefs, it is implausible, since equilibria in each framework is characterized by two asymmetric differential equations. Despite having analytical solutions for a special belief system, it cannot be done, as both the frameworks have different boundary conditions. In particular, both the boundary conditions cannot be true simultaneously.

[1]The first-price double auction (resp., second-price double auction) may be called as a *monopsony bargaining rule* (resp., *monopoly bargaining rule*).

Therefore, we compare them under symmetric beliefs. Particularly, we show that the bidders are better off under buyer-seller uncertainty than under buyer-seller certainty and no-trade situation.

1.1 The literature

To the best of my knowledge, the present paper is the first to consider that bidders' identities as a buyer and seller are determined by their bids for two indivisible objects. Our work fits in two strands of literature: double auctions and standard auctions.

Double auctions have been considered in Chatterjee and Samuelson [1], Myerson and Satterthwaite [11], Gresik [3, 4], Makowski and Mezzetti [9], Lu and Robert [8], Leininger et al. [7], Williams [13], among others. Standard auctions have been considered in Riley and Samuelson [12], Cheng [2], Kirkegaard [6], Maskin and Riley [10], Hu et al. [5], among others.

The double auction literature has largely considered that bidders' identities are exogenous. Chatterjee and Samuelson [1] characterize double auctions for one object with a predefined buyer and seller. Myerson and Satterthwaite [11] show that there does not exist any mechanism that is incentive compatible, interim individually rational and ex-post efficient. Gresik [4] generalizes Myerson and Satterthwaite's framework by considering ex-post individual rationality and shows that there is no loss in efficiency.

In some sense, our first-price double auction is analogous to the first-price auction while the second-price double auction is analogous to the second-price auction in single-unit standard auctions. Under symmetric auctions, Riley and Samuelson [12] show revenue equivalence principle between the first- and second price auction. Maskin and Riley [10] consider asymmetric auctions and show that a general revenue ranking principle between the first- and second-price auction does not exist.

2 The model

Consider a double auction with two indivisible objects that are complements to each other. There are two risk-neutral bidders who own one object each. Let the set of bidders be $N = \{1, 2\}$. The value of the object is privately known to the bidders. The value space is $T_i = [0, a_i] \subset \Re_+$ for every $i \in N$ with $a_1, a_2 \in \Re_{++}$.

Let the random variables of values be $\mathcal{T}_1$ and $\mathcal{T}_2$. The probability distribution is $F_i : T_i \to \Re_+$ with density functions $f_i : T_i \to \Re_+$ for every $i \in N$. We assume that the distribution functions are twice continuously differentiable and the density functions are atomless. Let the degree of complementarity between the two objects be denoted by $\mu \geq 1$. That is, if a bidder draws a value of t, then his value from owning his object is t and if he obtains the second object, his value rises to $(\mu+1)t$. Note that μ is homogeneous and common knowledge between the bidders.

We consider two economic environments: (a) buyer-seller uncertainty and (b) buyer-seller certainty. Under buyer-seller uncertainty, the identities of the buyer and seller are endogenous, i.e., they are determined by their bids. In particular, the highest bidder serves as a buyer while the lowest bidder serves as a seller. Under buyer-seller certainty, the identities of the buyer and seller are exogenous, i.e., they are predetermined before bids are placed. The buyer-seller uncertainty is considered in Section 3 and the buyer-seller certainty is considered in Section 4.

Let a family of bargaining rules be:

$$p(b_B, b_S) := \lambda_B b_B + \lambda_S b_S$$

where b_B is the buyer's bid, b_S is the seller's bid and λ_B and λ_S are exogenous parameters which indicate the market power of the buyer and seller respectively.

If $\lambda_B = 1$ and $\lambda_S = 0$, the bargaining rule is called the first-price double auction. If $\lambda_B = 0$ and $\lambda_S = 1$, the bargaining rule is called the second-price double auction. In the first-price double auction, the price of an object equals the highest bid while in the second-price double auction, the price of an object equals the lowest bid. That is, all the market power is allocated to the buyer in the first-price double auction and all the market power is allocated to the seller in the second-price double auction.

3 Buyer-seller uncertainty

In this section, we consider that the identities of the buyer and seller are uncertain while placing bids; they are revealed only after the bids are submitted. Each bidder is supposed to bid in a sealed envelope. Whosoever bids the highest acts as a buyer while the other bidder acts as a seller. If bidder i with value t_i bids b_i while bidder j bids b_j so that $b_i \geq b_j$, then bidder j's object is allocated to bidder i at a price of $\lambda_B b_i + \lambda_S b_j$. In this case, bidder i's ex-post utility is $(\mu+1)t_i - \lambda_B b_i - \lambda_S b_j$.

Let the bid functions be continuous, strictly monotone and onto. Let the bid function be denoted by σ_i and the inverse bid function be denoted by π_i for every $i \in N$. Consider bidder $i \in N$ with value t_i. Suppose he bids b while bidder j implements his bid function σ_j. Then, bidder i becomes a buyer if and only if $b \geq \sigma_j(\mathcal{T}_j)$, which equals $\mathcal{T}_j \leq \pi_j(b)$. Otherwise, he becomes a seller. Therefore, the expected utility function, $U_i : T_i \times \Re_+ \to \Re$, is

$$
\begin{aligned}
U_i(t_i, b) = &\int_0^{\pi_j(b)} [(\mu+1)t_i - \lambda_B b - \lambda_S \sigma_j(\omega)] f_j(\omega) \mathrm{d}\omega \\
&+ \int_{\pi_j(b)}^{a_j} [\lambda_B \sigma_j(\omega) + \lambda_S b - t_i] f_j(\omega) \mathrm{d}\omega
\end{aligned} \tag{1}
$$

where the first term is the expected utility from being a buyer while the second term is the expected utility from being a seller.

In the following result, we characterize an equilibrium.

Proposition 1. *A profile of measurable functions* (π_1, π_2) *constitutes a Bayesian equilibrium if and only if*

$$
\begin{aligned}
\mathrm{D}\pi_2(b) &= \frac{(\lambda_B + \lambda_S)F_2 \circ \pi_2(b) - \lambda_S}{f_2 \circ \pi_2(b)} \frac{1}{(\mu+2)\pi_1(b) - 2(\lambda_B + \lambda_S)b} \\
\mathrm{D}\pi_1(b) &= \frac{(\lambda_B + \lambda_S)F_1 \circ \pi_1(b) - \lambda_S}{f_1 \circ \pi_1(b)} \frac{1}{(\mu+2)\pi_2(b) - 2(\lambda_B + \lambda_S)b} \\
\pi_1(0) &= \pi_2(0) = 0, \quad \sigma_1(a_1) = \sigma_2(a_2) \quad a.e.
\end{aligned} \tag{2}
$$

We say that beliefs are symmetric if $F_1 = F_2 \triangleq F$ and $a_1 = a_2 \triangleq a$. Under a symmetric equilibrium, we denote the value spaces, values, inverse bid function, bid functions and utility functions by dropping the subscripts. We use superscripts to denote the first- and second-price double auction.

Let $P^B(\pi)$ be the *ex-ante* expected buy price, $P^S(\pi)$ be the *ex-ante* expected sell price and $V(\pi)$ be the bidders' *ex-ante* expected utility. Let V^* be the bidders' *ex-ante* expected utility under the no-trade situation.

We establish one of the main results below.

Theorem 1. *Let* π *be a symmetric Bayesian equilibrium. Then,*

1. $P^B(\pi) = P^S(\pi) \triangleq P(\pi)$
2. $V(\pi) = \int_0^a [(\mu+2)F(\omega) - 1]\omega f(\omega)\mathrm{d}\omega$
3. *As long as* $\mu > 2\int_0^a \omega f(\omega)\mathrm{d}\omega / \int_0^a \omega f(\omega)F(\omega)\mathrm{d}\omega - 2$, $V(\pi) > V^*$.

The first part is the price-equivalence principle which says that the *ex-ante* expected sell price equals *ex-ante* expected buy price. The second part is the payoff-equivalence principle which says that the impact of market power is redundant on the bidders' expected utilities. The third part says that the endogenous selection of buyer and seller dominates the no-trade situation, as long as the degree of complementarity is sufficiently large.

Consider the first-price double auction (resp., second-price double auction). For every $t \in T$, the ex-post buy price is lower (resp., higher) than the ex-post sell price. Given $t \in T$, the interim expected buy price is $\int_0^t \sigma(t)f(\omega)\mathrm{d}\omega$ (resp., $\int_0^t \sigma(\omega)f(\omega)\mathrm{d}\omega$) and interim expected sell price is $\int_t^a \sigma(\omega)f(\omega)\mathrm{d}\omega$ (resp., $\int_t^a \sigma(t)f(\omega)\mathrm{d}\omega$). For sufficiently low (resp., high) values, the interim expected sell price exceeds the interim expected buy price while for sufficiently high (resp., low) values, the interim expected buy price exceeds the interim expected sell price. Moreover, the interim expected buy price rises (resp., declines) in value while the interim expected sell price declines (resp., rises) in value.

The rest of the results of the present section concern about either the first-price double auction or the second-price double auction.

Proposition 2. *Let π^1 be a symmetric Bayesian equilibrium under the first-price double auction and let F/f be increasing everywhere on the type space. Then, $(\mu+1)\mathrm{D}\pi^1(b) > 1$ a.e.*

The above result interprets that if a bidder ends up being a buyer, then his utility rises in his value. From (1), the bidder's utility conditional on being a buyer is $(\mu+1)\pi^1(b) - b$. If $(\mu+1)\mathrm{D}\pi^1(b) > 1$, then $(\mu+1)\pi^1(b) - b$ is strictly increasing in b. Consequently, $(\mu+1)t - \sigma^1(t)$ rises in t. Since $\sigma^1(0) = 0$ and $(\mu+1)t - \sigma^1(t)$ rises in t, it must be the case that $(\mu+1)t - \sigma^1(t) > 0$ for every $t \in (0, a)$ which equalizes that

$$\sigma^1(t) < (\mu+1)t$$

for every $t \in (0, a)$.

Under symmetric equilibrium, an explicit expression for the bid function cannot be established. Nonetheless, we establish lower and upper bounds on the bids and prices under the first-price double auction.

Proposition 3. *Let π^1 be a symmetric Bayesian equilibrium under the first-price double auction.*

1. *For every $t \in [0, a]$, $\underline{\sigma}^1(t) < \sigma^1(t) < \bar{\sigma}^1(t)$ where*

$$\underline{\sigma}^1(t) = \frac{\mu+2}{2F(t)} \int_0^t \omega f(\omega)\mathrm{d}\omega$$

is the lower bound of bid by a bidder and

$$\bar{\sigma}^1(t) = (\mu+1)t$$

is the upper bound of bid by a bidder.

2. *$\underline{P}(\pi^1) < P(\pi^1) < \bar{P}(\pi^1)$ where*

$$\underline{P}(\pi^1) = \frac{\mu+2}{2} \int_0^a \omega f(\omega)[1 - F(\omega)]\mathrm{d}\omega$$

is the lower bound of ex-ante *expected price at which trade occurs and*

$$\bar{P}(\pi^1) = (\mu+2) \int_0^a \omega f(\omega)[1 - F(\omega)]\mathrm{d}\omega$$

is the upper bound of ex-ante *expected price at which trade occurs.*

In the result that follows, we compare the bid functions between the first-price double auction and second-price double auctions.

Proposition 4. *Let π^1 be a symmetric Bayesian equilibrium under the first-price double auction and let π^2 be a symmetric Bayesian equilibrium under the second-price double auction. Then, $\pi^2(b) > \pi^1(b)$ for every $b \in (0, \sigma^2(a))$.*

The above result conveys that the bidders bid higher under the first-price double auction than under the second-price double auction. The intuition is as follows. Under the first-price double auction, the buyer will have all the market power while under the second-price double auction, the seller will have all the market power. So, under the first-price double auction, a bidder will want to be a buyer while under the second-price double auction, a bidder will want to be a seller. As a result, a bidder bids higher under the first-price double auction and bids lower under the first-price double auction.

The next result compares the bid functions and bid distributions between the two bidders under the the first- and second-price double auction.

Proposition 5. *Let (π_1^1, π_2^1) be a Bayesian equilibrium under the first-price double auction. Let (π_1^2, π_2^2) be a Bayesian equilibrium under the second-price double auction. Then,*

1. *If $f_1/F_1 > f_2/F_2$ with $a_1 > a_2$, then $\pi_1^1(b) > \pi_2^1(b)$ and $F_1 \circ \pi_1^1(b) < F_2 \circ \pi_2^1(b)$ for every $b \in (0, \sigma_1^1(a_1))$.*
2. *If $f_1/(1-F_1) > f_2/(1-F_2)$ with $a_1 > a_2$, then $\pi_1^2(b) > \pi_2^2(b)$ and $F_1 \circ \pi_1^2(b) < F_2 \circ \pi_2^2(b)$ for every $b \in (0, \sigma_1^2(a_1))$.*

The above result says that, bidder 1 bids lower and produces a stronger bid distribution than bidder 2 as long as bidder 1's probability distribution dominates bidder 2's probability distribution. As a result, bidder 1's probability of being a buyer is higher than that of bidder 2.

4 Buyer-seller certainty

In this section, we consider that bidders know the identities of buyer and seller before they place their bids, i.e., the identities are exogenously determined.

We restrict to the family of continuous, strictly monotone and onto bid functions. Let the bid function of player $i \in N$ be θ_i and the inverse bid function be ϕ_i. Since we do not impose any stochastic order on the probability distributions, without loss of generality, *bidder 1 is the buyer while bidder 2 is the seller.* Thus, trade takes place if and only if the buyer's bid weakly exceeds the seller's bid.

Consider bidder 1 with value t_1 and bid b. Let bidder 2 respect θ_2. Then, trade takes place if and only if $b \geq \theta_2(\mathcal{T}_2)$ which is equivalent to $\mathcal{T}_2 \leq \phi_2(b)$. Therefore, the expected utility of bidder 1 is

$$U_1(t_1, b) = \int_0^{\phi_2(b)} [(\mu+1)t_1 - \lambda_B b - \lambda_S \theta_2(\omega)] f_2(\omega) \mathrm{d}\omega + [1 - F_2 \circ \phi_2(b)] t_1 \quad (3)$$

where the first term indicates the expected utility in case the trade succeeds and the second term indicates the expected utility in case the trade fails.

Now, consider bidder 2 with value t_2 and bid b. Let bidder 1 respect θ_1. Then, trade takes place if and only if $b \leq \theta_1(\mathcal{T}_1)$ which is equivalent to $\mathcal{T}_1 \geq \phi_1(b)$. Therefore, the expected utility of bidder 2 is

$$U_2(t_2, b) = \int_{\phi_1(b)}^{a_1} [\lambda_B \theta_1(\omega) + \lambda_S b - t_2] f_1(\omega) \mathrm{d}\omega + F_1 \circ \phi_1(b) t_2 \quad (4)$$

where the first term indicates the expected utility in case the trade succeeds and the second term indicates the expected utility in case the trade fails.

In the following result, we characterize an equilibrium.

Proposition 6. *A profile of measurable functions (ϕ_1, ϕ_2) constitutes a Bayesian equilibrium if and only if*

$$\begin{aligned} \mathrm{D}\phi_2(b) &= \frac{\lambda_B F_2 \circ \phi_2(b)}{f_2 \circ \phi_2(b)} \frac{1}{\mu\phi_1(b) - (\lambda_B + \lambda_S) b} \\ \mathrm{D}\phi_1(b) &= \frac{\lambda_S [1 - F_1 \circ \phi_1(b)]}{f_1 \circ \phi_1(b)} \frac{1}{(\lambda_B + \lambda_S) b - 2\phi_2(b)} \\ \phi_1(0) &= \phi_2(0) = 0, \quad \theta_1(a_1) = \theta_2(a_2) \quad a.e. \end{aligned} \quad (5)$$

Remark 1. *Under the first-price double auction, the equilibrium is characterized as*

$$\phi_1^1(b) = \frac{b}{\mu} + \frac{2}{\mu} \frac{F_2(b/2)}{f_2(b/2)}, \quad \phi_2^1(b) = \frac{b}{2} \quad (6)$$

Under the second-price double auction, the equilibrium is characterized as

$$\phi_1^2(b) = \frac{b}{\mu}, \quad \phi_2^2(b) = \frac{b}{2} - \frac{\mu}{2} \frac{1 - F_1(b/\mu)}{f_1(b/\mu)} \quad (7)$$

The next result presents a comparative property.

Proposition 7. *Let (ϕ_1^1, ϕ_2^1) be a Bayesian equilibrium under the first-price double auction. Let (ϕ_1^2, ϕ_2^2) be a Bayesian equilibrium under the second-price double auction. Then,*

1. $\phi_1^1(b) > \phi_2^1(b)$ *and* $\phi_1^2(b) > \phi_2^2(b)$ *a.e. as long as* $\mu \leq 2$.
2. $\phi_1^1(b) > \phi_1^2(b)$ *and* $\phi_2^1(b) > \phi_2^2(b)$ *a.e.*
3. $F_2 \circ \phi_2^1(b) < F_1 \circ \phi_1^1(b)$ *a.e. as long as* $\mu \leq 2$ *and* $F_2 < F_1$.
4. $F_2 \circ \phi_2^2(b) < F_1 \circ \phi_1^2(b)$ *a.e. as long as* $\mu \leq 2$ *and* $F_2 < F_1$.

The first part says that the seller bids higher than the buyer as long as the degree of complementarity is sufficiently low. The second part says that both the buyer and seller bid higher under the second-price double auction than under the first-price double auction. The third and fourth parts say that the seller produces a stronger bid distribution than the buyer as long as the degree of complementarity is sufficiently low and the seller's probability distribution dominates the buyer's, i.e., buyer's belief about trade occurring is less than the seller's belief about the trade failing.

5 A special belief system

In this section, we do a comparative analysis of expected prices, expected probabilities and expected utilities for a special family of power distributions. Consider the first-price double auction. In the buyer-seller uncertainty, let $P_i^B(\pi_1^1, \pi_2^1)$ be bidder i's *ex-ante* expected buy price, $P_i^S(\pi_1^1, \pi_2^1)$ be bidder i's *ex-ante* expected sell price and $C_i(\pi_1^1, \pi_2^1)$ be bidder i's *ex-ante* expected probability of being a buyer for every $i \in N$. In the buyer-seller certainty, let $P_1(\phi_1^1, \phi_2^1)$ be the *ex-ante* expected buy price, $P_2(\phi_1^1, \phi_2^1)$ be the *ex-ante* expected sell price and $C_i(\phi_1^1, \phi_2^1)$ be the *ex-ante* expected probability of bidder i for every $i \in N$.

Define a family of probability distributions as

$$\mathcal{F} := \left\{ F_i : T_i \to \Re_+ \middle| F_i(t_i) = \left(\frac{t_i}{a_i}\right)^{\kappa_i} \quad \forall i \in N \right\} \tag{8}$$

where $\kappa_1, \kappa_2 > 0$.

In the following result, we compare the expected prices and expected probabilities between the two bidders under the buyer-seller uncertainty.

Theorem 2. *Let* (π_1^1, π_2^1) *be a Bayesian equilibrium of the buyer-seller uncertainty environment under the first-price double auction. Let* $F_1, F_2 \in \mathcal{F}$, $\kappa_2 a_1(2\kappa_1 + 1) = \kappa_1 a_2(2\kappa_2 + 1)$ *and* $\kappa_1 > \kappa_2$. *Then,*
1. $P_1^B(\pi_1^1, \pi_2^1) = P_2^S(\pi_1^1, \pi_2^1) > P_2^B(\pi_1^1, \pi_2^1) = P_1^S(\pi_1^1, \pi_2^1)$
2. $C_1(\pi_1^1, \pi_2^1) > C_2(\pi_1^1, \pi_2^1)$

It says that, if bidder 1's probability distribution dominates bidder 2's probability distribution, then (a) bidder 1's expected buy price equals bidder 2's expected sell price, (b) bidder 1's expected sell price equals bidder 2's expected buy price, (c) bidder 1's expected buy price exceeds his expected sell price, (d) bidder 2's expected sell price exceeds his expected buy price, and (e) bidder 1's expected probability of being a buyer exceeds that of bidder 2's.

In the next result, we compare the expected prices and expected probabilities of the buyer and seller under the buyer-seller certainty.

Theorem 3. *Let (ϕ_1^1, ϕ_2^1) be a Bayesian equilibrium of the buyer-seller certainty under the first-price double auction. Let $F_1, F_2 \in \mathcal{F}$ and $2a_2(1+\kappa_2) = \mu\kappa_2 a_1$. Then,*

1. $P_1(\phi_1^1, \phi_2^1) = P_2(\phi_1^1, \phi_2^1)$
2. $C_1(\phi_1^1, \phi_2^1) = C_2(\phi_1^1, \phi_2^1)$

The above result establishes (a) price-equivalence and (b) probability-equivalence. Despite the asymmetry among bidders, the expected buy price equals the expected sell price and expected probability of trade occurring is equivalent for the buyer and seller. Note that no stochastic order on probability distributions is imposed to establish the aforesaid property.

The next result compares the bid functions and expected prices between the buyer-seller uncertainty and buyer-seller certainty environments.

Theorem 4. *Let π^1 be a symmetric Bayesian equilibrium of the buyer-seller uncertainty under the first-price double auction. Let ϕ^1 be a symmetric Bayesian equilibrium of the buyer-seller certainty under the first-price double auction. Let beliefs be symmetric, i.e., $a_1 = a_2 \triangleq 1$ and $\kappa_1 = \kappa_2 \triangleq \kappa$. Let $\mu = 2(1+\kappa)/\kappa$. Then,*

1. *$\sigma^1(t) = \theta^1(t)$ for every $t \in [0,1]$.*
2. $P(\pi^1) = P_1(\phi^1) = P_2(\phi^1)$
3. $V(\pi^1) > V_1(\phi^1) > V_2(\phi^1) > V^*$

The above result establishes certain properties: (a) bids are equivalent under buyer-seller uncertainty and buyer-seller certainty, (b) the *ex-ante* expected prices at which trade occurs are equivalent under buyer-seller uncertainty and under buyer-seller certainty, (c) buyer-seller uncertainty dominates buyer-seller certainty and no-trade situation.

6 Conclusion

In this paper, we have considered a double auction with two bidders and two objects that are complements. Each bidder owns exactly one object. Since the two objects are complements, bidders can benefit from trading with each other. If the identities of buyer and seller are determined before they place bids, the trade may lead to inefficiency in one of the following ways: (a) both the bidders wants to be buyers, (b) both wants to be sellers, and (c) one wants to be buyer while the other wants to be seller. In order to overcome the inefficiency, we have considered that the identities of the buyer and seller are determined by the bids.

Under symmetric beliefs, we have shown that the sell price equals the buy price in expectations and the bidders' expected utilities are equivalent under all bargaining rules. Under asymmetric beliefs, the expected buy and sell prices may be unequal. We have also shown that, for a special belief system, bidders' are better off by endogenous selection of the buyer and seller's identities.

Appendix A: Proofs

Proof of Proposition 1. Let (π_1, π_2) be a Bayesian equilibrium. Consider bidder i with value t_i and bid b. Using Leibniz integral rule, the first-order derivative of (1) is

$$\begin{aligned} \mathrm{D}_b U_i(t_i, b) = {} & \mathrm{D}_b F_j \circ \pi_j(b)[(\mu+1)t_i - (\lambda_B + \lambda_S)b] - \lambda_B F_j \circ \pi_j(b) \\ & - \mathrm{D}_b F_j \circ \pi_j(b)[(\lambda_B + \lambda_S)b - t_i] + \lambda_S[1 - F_j \circ \pi_j(b)] \end{aligned}$$

In equilibrium, $t_i = \pi_i(b)$ and $\mathrm{D}_b U_i(\pi_i(b), b) = 0$. Therefore, we have

$$\mathrm{D}_b F_j \circ \pi_j(b)[(\mu+2)\pi_i(b) - 2(\lambda_B + \lambda_S)b] = (\lambda_B + \lambda_S) F_j \circ \pi_j(b) - \lambda_S$$

which simplifies to the system of differential equations in (2).

To show the converse, let a profile of measurable functions (π_1, π_2) satisfies the Dirichlet problem in (2). Consider bidder i with value t_i and bid b. Suppose he overbids to c so that $\pi_i(c) > t_i$. Then, from the first-order derivative of (1), we have

$$\begin{aligned} \mathrm{D}_c U_i(t_i, c) &= \mathrm{D}_c F_j \circ \pi_j(c)[(\mu+2)t_i - 2(\lambda_B + \lambda_S)c] + \lambda_S \\ &\quad - (\lambda_B + \lambda_S) F_j \circ \pi_j(c) \\ &< \mathrm{D}_c F_j \circ \pi_j(c)[(\mu+2)\pi_i(c) - 2(\lambda_B + \lambda_S)c] + \lambda_S \\ &\quad - (\lambda_B + \lambda_S) F_j \circ \pi_j(c) \\ &= \mathrm{D}_c U_i(\pi_i(c), c) \\ &= 0 \end{aligned}$$

which signifies that overbids reduce the expected utility. In similar vein, it can be shown that underbids raise the expected utility. Thus, (π_1, π_2) is stable. ■

Proof of Theorem 1. We show the first part. Consider a bidder with value t. Then, his *ex-ante* expected buy price is

$$
\begin{aligned}
P^B(\pi) &= \int_0^a \left\{ \int_0^t [\lambda_B\sigma(t) + \lambda_S\sigma(\omega)] f(\omega)\mathrm{d}\omega \right\} f(t)\mathrm{d}t \\
&= \lambda_B \int_0^a \int_0^t \sigma(t) f(\omega) f(t)\mathrm{d}\omega\mathrm{d}t + \lambda_S \int_0^a \int_0^t \sigma(\omega) f(\omega) f(t)\mathrm{d}\omega\mathrm{d}t
\end{aligned}
$$

Using Fubini's theorem, we have

$$
\begin{aligned}
P^B(\pi) &= \lambda_B \int_0^a \sigma(t) f(t) F(t)\mathrm{d}t + \lambda_S \int_0^a \int_t^a \sigma(t) f(t) f(\omega)\mathrm{d}\omega\mathrm{d}t \\
&= \lambda_B \int_0^a \sigma(t) f(t) F(t)\mathrm{d}t + \lambda_S \int_0^a \sigma(t) f(t)[1 - F(t)]\mathrm{d}t \\
&= \int_0^a [\lambda_S + (\lambda_B - \lambda_S)F(t)]\sigma(t) f(t)\mathrm{d}t
\end{aligned}
$$

His *ex-ante* expected sell price is

$$
\begin{aligned}
P^S(\pi) &= \int_0^a \left\{ \int_t^a [\lambda_B\sigma(\omega) + \lambda_S\sigma(t)] f(\omega)\mathrm{d}\omega \right\} f(t)\mathrm{d}t \\
&= \lambda_B \int_0^a \int_t^a \sigma(\omega) f(\omega) f(t)\mathrm{d}\omega\mathrm{d}t + \lambda_S \int_0^a \int_t^a \sigma(t) f(\omega) f(t)\mathrm{d}\omega\mathrm{d}t
\end{aligned}
$$

Using Fubini's theorem, we have

$$
\begin{aligned}
P^S(\pi) &= \lambda_B \int_0^a \int_0^t \sigma(t) f(t) f(\omega)\mathrm{d}\omega\mathrm{d}t + \lambda_S \int_0^a \sigma(t) f(t)[1 - F(t)]\mathrm{d}t \\
&= \lambda_B \int_0^a \sigma(t) f(t) F(t)\mathrm{d}t + \lambda_S \int_0^a \sigma(t) f(t)[1 - F(t)]\mathrm{d}t \\
&= \int_0^a [\lambda_S + (\lambda_B - \lambda_S)F(t)]\sigma(t) f(t)\mathrm{d}t \\
&= P^B(\pi)
\end{aligned}
$$

Therefore, the price-equivalence principle holds.

We show the second part. Consider a bidder with value t. Then, his interim expected utility is

$$
\begin{aligned}
U(t, \sigma(t)) &= \int_0^t [(\mu + 1)t - \lambda_B\sigma(t) - \lambda_S\sigma(\omega)] f(\omega)\mathrm{d}\omega \\
&\quad + \int_t^a [\lambda_B\sigma(\omega) + \lambda_S\sigma(t) - t] f(\omega)\mathrm{d}\omega \\
&= [(\mu + 2)F(t) - 1]t - \int_0^t [\lambda_B\sigma(t) + \lambda_S\sigma(\omega)] f(\omega)\mathrm{d}\omega \\
&\quad + \int_t^a [\lambda_B\sigma(\omega) + \lambda_S\sigma(t)] f(\omega)\mathrm{d}\omega
\end{aligned}
$$

Therefore, the *ex-ante* expected utility is

$$\begin{aligned}V(\pi) &= \int_0^a U(t,\sigma(t))f(t)\mathrm{d}t \\ &= \int_0^a [(\mu+2)F(t)-1]tf(t)\mathrm{d}t \\ &\quad - \int_0^a \int_0^t [\lambda_B\sigma(t)+\lambda_S\sigma(\omega)]f(\omega)f(t)\mathrm{d}\omega\mathrm{d}t \\ &\quad + \int_0^a \int_t^a [\lambda_B\sigma(\omega)+\lambda_S\sigma(t)]f(\omega)f(t)\mathrm{d}\omega\mathrm{d}t \\ &= \int_0^a [(\mu+2)F(t)-1]tf(t)\mathrm{d}t - P^B(\pi) + P^S(\pi)\end{aligned}$$

Since $P^B(\pi) = P^S(\pi)$, we have

$$V(\pi) = \int_0^a [(\mu+2)F(t)-1]tf(t)\mathrm{d}t$$

which establishes the payoff-equivalence principle.

We show the third part. Under no-trade situation, the *ex-ante* expected payoff is

$$V^* = \int_0^a \omega f(\omega)\mathrm{d}\omega$$

Comparing $V(\pi)$ with V^*, we get the desired result. ■

Proof of Proposition 2. Consider bidder i with value t and bid b. In equilibrium, $t = \pi^1(b)$. We show that $(\mu+1)\mathrm{D}\pi^1(b) > 1$ for every $b \in (0, \sigma^1(a))$. From (2), we have

$$\begin{aligned}&\lim_{b\downarrow 0} \mathrm{D}\pi^1(b) = \lim_{b\downarrow 0} \frac{F\circ\pi^1(b)}{f\circ\pi^1(b)} \frac{1}{(\mu+2)\pi^1(b)-2b} \\ &= \lim_{b\downarrow 0} \frac{\mathrm{D}\pi^1(b)f\circ\pi^1(b)}{f\circ\pi^1(b)[(\mu+2)\mathrm{D}\pi^1(b)-2] + \mathrm{D}[f\circ\pi^1(b)][(\mu+2)\pi^1(b)-2b]}\end{aligned}$$

which implies

$$\mathrm{D}\pi^1(0) = \frac{3}{\mu+2}$$

Therefore, we have

$$(\mu+1)\mathrm{D}\pi^1(0) - 1 = \frac{2\mu+1}{\mu+2} > 0$$

Since π^1 is continuous, it follows that there exists $\epsilon > 0$ sufficiently close to 0 so that $(\mu+1)\mathrm{D}\pi^1(\epsilon) > 1+\epsilon$. We show that there does not exist

b so that $(\mu+1)\mathrm{D}\pi^1(b)=1$. To do so, we contradict. Suppose there exists $b^* \in (0,\sigma^1(a))$ so that $(\mu+1)\mathrm{D}\pi^1(b^*)=1$ and $(\mu+1)\mathrm{D}\pi^1(b)>1$ for every $b\in(0,b^*)$. Then, from (2), the second-order derivative is

$$\begin{aligned}
\mathrm{D}^2\pi^1(b^*) &= -\frac{F\circ\pi^1(b^*)}{f\circ\pi^1(b^*)}\frac{(\mu+2)\mathrm{D}\pi^1(b^*)-2}{[(\mu+2)\pi^1(b^*)-2b^*]^2} \\
&\quad+\frac{1}{(\mu+2)\pi^1(b^*)-2b^*}\mathrm{D}\left[\frac{F\circ\pi^1(b^*)}{f\circ\pi^1(b^*)}\right] \\
&=\frac{F\circ\pi^1(b^*)}{f\circ\pi^1(b^*)}\frac{\mu}{(\mu+1)[(\mu+2)\pi^1(b^*)-2b^*]^2} \\
&\quad+\frac{1}{(\mu+2)\pi^1(b^*)-2b^*}\mathrm{D}\left[\frac{F\circ\pi^1(b^*)}{f\circ\pi^1(b^*)}\right] \\
&>0
\end{aligned}$$

as F/f is strictly increasing. This implies $(\mu+1)\mathrm{D}^2\pi^1(b^*)>0$. Therefore, there exists $\delta>0$ so that $(\mu+1)\mathrm{D}\pi^1(b^*-\delta)<1-\delta$, which establishes a contradiction. Thus, $(\mu+1)\mathrm{D}\pi^1(b)>1$ for every $b\in(0,\sigma^1(a))$. ■

Proof of Proposition 3. We show the first part. From (2), we have

$$\mathrm{D}\pi^1(b)=\frac{F\circ\pi^1(b)}{f\circ\pi^1(b)}\frac{1}{(\mu+2)\pi^1(b)-2b}$$

Since $b=\sigma^1\circ\pi^1(b)$ implies $1=\mathrm{D}\sigma^1\circ\pi^1(b)\mathrm{D}\pi^1(b)$, we have

$$\mathrm{D}\sigma^1\circ\pi^1(b)=\frac{f\circ\pi^1(b)}{F\circ\pi^1(b)}[(\mu+2)\pi^1(b)-2b]$$

which equals

$$\mathrm{D}\sigma^1(t)=\frac{f(t)}{F(t)}[(\mu+2)t-2\sigma^1(t)]$$

as $t=\pi^1(b)$. This can be rewritten as

$$\mathrm{D}[F(t)\sigma^1(t)]=(\mu+2)tf(t)-\sigma^1(t)f(t)$$

Using the fundamental theorem of calculus on limits t and 0 along with $\sigma^1(0)=0$, we have

$$\begin{aligned}
\sigma^1(t) &= \frac{\mu+2}{F(t)}\int_0^t \omega f(\omega)\mathrm{d}\omega-\frac{1}{F(t)}\int_0^t\sigma^1(\omega)f(\omega)\mathrm{d}\omega \\
&>\frac{\mu+2}{F(t)}\int_0^t \omega f(\omega)\mathrm{d}\omega-\frac{1}{F(t)}\int_0^t\sigma^1(t)f(\omega)\mathrm{d}\omega \\
&=\frac{\mu+2}{F(t)}\int_0^t \omega f(\omega)\mathrm{d}\omega-\sigma^1(t)
\end{aligned}$$

which implies

$$\sigma^1(t) > \frac{\mu+2}{2F(t)} \int_0^t \omega f(\omega) \mathrm{d}\omega$$

Therefore, we have established a lower bound of $\sigma^1(t)$ for every $t \in T$. The upper bound has already been established in the main body of the paper.

We show the second part. The *ex-ante* expected price is

$$\begin{aligned} P(\pi^1) &= \int_0^a F(t)\sigma^1(t)f(t)\mathrm{d}t \\ &> \frac{\mu+2}{2} \int_0^a \int_0^t \omega f(\omega)f(t)\mathrm{d}\omega\mathrm{d}t \\ &= \frac{\mu+2}{2} \int_0^a \int_t^a tf(t)f(\omega)\mathrm{d}\omega\mathrm{d}t \\ &= \frac{\mu+2}{2} \int_0^a tf(t)[1-F(t)]\mathrm{d}t \end{aligned}$$

which gives a lower bound on the expected price. To establish an upper bound, the expected price can be written in terms of bid as

$$\begin{aligned} P(\pi^1) &= \int_0^{\sigma^1(a)} bF\circ\pi^1(b)F\circ\pi^1(\mathrm{d}b) \\ &= [bF\circ\pi^1(b)F\circ\pi^1(b)]_0^{\sigma^1(a)} - \int_0^{\sigma^1(a)} F\circ\pi^1(b)\mathrm{D}[bF\circ\pi^1(b)]\mathrm{d}b \\ &= \sigma^1(a) - \int_0^{\sigma^1(a)} F\circ\pi^1(b)\mathrm{D}[bF\circ\pi^1(b)]\mathrm{d}b \\ &= \int_0^{\sigma^1(a)} [1-F\circ\pi^1(b)]\mathrm{D}[bF\circ\pi^1(b)]\mathrm{d}b \end{aligned}$$

where the second equality arrives by applying integration-by-parts and the fourth equality arrives by using the condition $\sigma^1(a) = \int_0^{\sigma^1(a)} \mathrm{D}[bF\circ\pi^1(b)]\mathrm{d}b$. From (2), we have $\mathrm{D}[bF\circ\pi^1(b)]\mathrm{d}b = \mathrm{D}F\circ\pi^1(b)[(\mu+2)\pi^1(b)-b]$. Using it in the expression of $P(\pi^1)$, we have

$$P(\pi^1) = \int_0^{\sigma^1(a)} [1-F\circ\pi^1(b)]\mathrm{D}F\circ\pi^1(b)[(\mu+2)\pi^1(b)-b]\mathrm{d}b$$

Since $t = \pi^1(b)$, we have

$$\begin{aligned} P(\pi^1) &= \int_0^a [1-F(t)]f(t)[(\mu+2)t - \sigma^1(t)]\mathrm{d}t \\ &< (\mu+2)\int_0^a tf(t)[1-F(t)]\mathrm{d}t \end{aligned}$$

This establishes an upper bound on the expected price. ∎

Proof of Proposition 4. From (2) with the second-price double auction, we infer that

$$\mathrm{D}\sigma^2(t) < \frac{f(t)}{F(t)}[(\mu+2)t - 2\sigma^2(t)]$$

From (2) with the first-price double auction, we infer that

$$\mathrm{D}\sigma^1(t) = \frac{f(t)}{F(t)}[(\mu+2)t - 2\sigma^1(t)]$$

Suppose for some $t > 0$, $\sigma^2(t) \geq \sigma^1(t)$. Then, from the above two equations $\mathrm{D}\sigma^2(t) < \mathrm{D}\sigma^1(t)$. Since $\sigma^1(0) = \sigma^2(0) = 0$ and the two bid functions are continuous, $\sigma^2(t) \geq \sigma^1(t)$ implies $\mathrm{D}\sigma^2(t) < \mathrm{D}\sigma^1(t)$ leads to a contradiction. Therefore, for every $t > 0$, it should be the case that $\sigma^1(t) > \sigma^2(t)$. ∎

Proof of Proposition 5. We show the first part. As $\pi_1^1 \circ \sigma_1^1(a_1) = a_1 > a_2 = \pi_2^1 \circ \sigma_1^1(a_1)$, there exists $\epsilon > 0$ so that $\pi_1^1 \circ (\sigma_1^1(a_1) - \epsilon) > \pi_2^1 \circ (\sigma_1^1(a_1) - \epsilon)$. Suppose there exists $b^* \in (0, \sigma_1^1(a_1))$ so that $\pi_1^1(b^*) = \pi_2^1(b^*)$ and $\pi_1^1(b) > \pi_2^1(b)$ for every $b \in (b^*, \sigma_1^1(a_1))$. Then, from (2), we have

$$\begin{aligned}
\mathrm{D}\pi_2^1(b^*) &= \frac{F_2 \circ \pi_2^1(b^*)}{f_2 \circ \pi_2^1(b^*)} \frac{1}{(\mu+2)\pi_1^1(b^*) - 2b^*} \\
&= \frac{F_2 \circ \pi_1^1(b^*)}{f_2 \circ \pi_1^1(b^*)} \frac{1}{(\mu+2)\pi_2^1(b^*) - 2b^*} \\
&> \frac{F_1 \circ \pi_1^1(b^*)}{f_1 \circ \pi_1^1(b^*)} \frac{1}{(\mu+2)\pi_2^1(b^*) - 2b^*} \\
&= \mathrm{D}\pi_1^1(b^*)
\end{aligned}$$

Thus, there exists $\delta > 0$ so that $\pi_2^1(b^* + \delta) > \pi_1^1(b^* + \delta)$ – a contradiction. Hence, $\pi_1^1(b) > \pi_2^1(b)$ for every $b \in (0, \sigma_1^1(a_1))$.

From (2) with the first-price double auction and the result that $\pi_1^1(b) > \pi_2^1(b)$ for every $b \in (0, \sigma_1^1(a_1))$, we have

$$\frac{F_2 \circ \pi_2^1(b)}{\mathrm{D}F_2 \circ \pi_2^1(b)} = (\mu+2)\pi_1^1(b) - 2b > (\mu+2)\pi_2^1(b) - 2b = \frac{F_1 \circ \pi_1^1(b)}{\mathrm{D}F_1 \circ \pi_1^1(b)}$$

which implies

$$\mathrm{D}\left[\frac{F_1 \circ \pi_1^1(b)}{F_2 \circ \pi_2^1(b)}\right] > 0$$

Since $F_1 \circ \pi_1^1 \circ \sigma_1^1(a_1) = F_2 \circ \pi_2^1 \circ \sigma_1^1(a_1) = 1$ and the ratio is increasing, it must be the case that $F_1 \circ \pi_1^1(b) < F_2 \circ \pi_2^1(b)$ for every $b \in (0, \sigma_1^1(a_1))$.

We show the second part. As $\pi_1^2 \circ \sigma_1^2(a_1) = a_1 > a_2 = \pi_2^2 \circ \sigma_1^2(a_1)$, there exists $\epsilon > 0$ so that $\pi_1^2 \circ (\sigma_1^2(a_1) - \epsilon) > \pi_2^2 \circ (\sigma_1^2(a_1) - \epsilon)$. Suppose there exists $b^* \in (0, \sigma_1^2(a_1))$ so that $\pi_1^2(b^*) = \pi_2^2(b^*)$ and $\pi_1^2(b) > \pi_2^2(b)$ for every $b \in (b^*, \sigma_1^2(a_1))$. Then, from (2), we have

$$\begin{aligned}
\mathrm{D}\pi_2^2(b^*) &= \frac{1 - F_2 \circ \pi_2^2(b^*)}{f_2 \circ \pi_2^2(b^*)} \frac{1}{2b^* - (\mu + 2)\pi_1^2(b^*)} \\
&= \frac{1 - F_2 \circ \pi_1^2(b^*)}{f_2 \circ \pi_1^2(b^*)} \frac{1}{2b^* - (\mu + 2)\pi_2^2(b^*)} \\
&> \frac{1 - F_1 \circ \pi_1^2(b^*)}{f_1 \circ \pi_1^2(b^*)} \frac{1}{2b^* - (\mu + 2)\pi_2^2(b^*)} \\
&= \mathrm{D}\pi_1^2(b^*)
\end{aligned}$$

Thus, there exists $\delta > 0$ so that $\pi_2^2(b^* + \delta) > \pi_1^2(b^* + \delta)$ – a contradiction. Hence, $\pi_1^2(b) > \pi_2^2(b)$ for every $b \in (0, \sigma_1^2(a_1))$.

From (2) with the second-price double auction and the result that $\pi_1^2(b) > \pi_2^2(b)$ for every $b \in (0, \sigma_1^2(a_1))$, we have

$$\frac{1 - F_2 \circ \pi_2^2(b)}{\mathrm{D}F_2 \circ \pi_2^2(b)} = 2b - (\mu + 2)\pi_1^2(b) < 2b - (\mu + 2)\pi_2^2(b) = \frac{1 - F_1 \circ \pi_1^2(b)}{\mathrm{D}F_1 \circ \pi_1^2(b)}$$

which implies

$$\mathrm{D}\left[\frac{1 - F_2 \circ \pi_2^2(b)}{1 - F_1 \circ \pi_1^2(b)}\right] < 0$$

Since $1 - F_1 \circ \pi_1^2(0) = 1 - F_2 \circ \pi_2^2(0) = 1$ and the ratio is decreasing, it must be the case that $1 - F_2 \circ \pi_2^2(b) < 1 - F_1 \circ \pi_1^2(b)$ which is equivalent to $F_1 \circ \pi_1^2(b) < F_2 \circ \pi_2^2(b)$ for every $b \in (0, \sigma_1^2(a_1))$. ■

Proof of Proposition 6. Let (ϕ_1, ϕ_2) be a Bayesian equilibrium. Using Leibniz integral rule, the first-order derivative of (3) gives

$$\mathrm{D}_b U_1(t_1, b) = \mathrm{D}\phi_2(b) f_2 \circ \phi_2(b)[\mu t_1 - (\lambda_B + \lambda_S)b] - \lambda_B F_2 \circ \phi_2(b)$$

In equilibrium, $t_1 = \phi_1(b)$ and utility is maximized when $\mathrm{D}_b U_1(\phi_1(b), b) = 0$. This gives us the first equation.

Using Leibniz integral rule, the first-order derivative of (4) gives

$$\mathrm{D}_b U_2(t_2, b) = \mathrm{D}\phi_1(b) f_1 \circ \phi_1(b)[2t_2 - (\lambda_B + \lambda_S)b] + \lambda_S[1 - F_1 \circ \phi_1(b)]$$

In equilibrium, $t_2 = \phi_2(b)$ and utility is maximized when $\mathrm{D}_b U_1(\phi_1(b), b) = 0$. This gives us the second equation.

We now show sufficiency. Let (ϕ_1, ϕ_2) solve (5). Consider bidder 1 with value t_1 and bid b so that $t_1 = \phi_1(b)$. Suppose he overbids to c so that $\phi_1(c) > t_1$. Then, the first-order derivative of (3) implies

$$\begin{aligned}
\mathrm{D}_c U_1(t_1, c) &= \mathrm{D}\phi_2(c) f_2 \circ \phi_2(c)[\mu t_1 - (\lambda_B + \lambda_S)c] - \lambda_B F_2 \circ \phi_2(c) \\
&< \mathrm{D}\phi_2(c) f_2 \circ \phi_2(c)[\mu \phi_1(c) - (\lambda_B + \lambda_S)c] - \lambda_B F_2 \circ \phi_2(c) \\
&= 0
\end{aligned}$$

which implies that overbids reduce expected utility. Simply by flipping the inequalities, we can show that underbids raise expected utility.

Now, consider bidder 2 with value t_2 and bid b so that $t_2 = \phi_2(b)$. Suppose he overbids to c so that $\phi_2(c) > t_2$. Then, the first-order derivative of (4) implies

$$\begin{aligned}
\mathrm{D}_c U_2(t_2, c) &= \mathrm{D}\phi_1(c) f_1 \circ \phi_1(c)[2t_2 - (\lambda_B + \lambda_S)c] + \lambda_S[1 - F_1 \circ \phi_1(c)] \\
&< \mathrm{D}\phi_1(c) f_1 \circ \phi_1(c)[2\phi_2(c) - (\lambda_B + \lambda_S)c] + \lambda_S[1 - F_1 \circ \phi_1(c)] \\
&= 0
\end{aligned}$$

which implies that overbids reduce expected utility. Simply by flipping the inequalities, we can show that underbids raise expected utility. Thus, (ϕ_1, ϕ_2) is an equilibrium. ■

Proof of Theorem 2. From (2), we have the inverse bid functions as

$$\pi_1^1(b) = \frac{2\kappa_2 + 1}{\kappa_2(\mu + 2)} b, \quad \pi_2^1(b) = \frac{2\kappa_1 + 1}{\kappa_1(\mu + 2)} b$$

and bid functions as

$$\sigma_1^1(t_1) = \frac{\kappa_2(\mu + 2)}{2\kappa_2 + 1} t_1, \quad \sigma_2^1(t_2) = \frac{\kappa_1(\mu + 2)}{2\kappa_1 + 1} t_2$$

The least upper bound of bids is

$$\sup b \triangleq \sigma_1^1(a_1) = \frac{a_1 \kappa_2(\mu + 2)}{2\kappa_2 + 1} = \frac{a_2 \kappa_1(\mu + 2)}{2\kappa_1 + 1} = \sigma_2^1(a_2)$$

The bid distributions are

$$F_1 \circ \pi_1^1(b) = \left[\frac{2\kappa_2 + 1}{a_1 \kappa_2(\mu + 2)}\right]^{\kappa_1} b^{\kappa_1}, \quad F_2 \circ \pi_2^1(b) = \left[\frac{2\kappa_1 + 1}{a_2 \kappa_1(\mu + 2)}\right]^{\kappa_2} b^{\kappa_2}$$

and their associated densities are

$$\mathrm{D}F_1 \circ \pi_1^1(b) = \kappa_1 \left[\frac{2\kappa_2 + 1}{a_1\kappa_2(\mu+2)}\right]^{\kappa_1} b^{\kappa_1 - 1}$$
$$\mathrm{D}F_2 \circ \pi_2^1(b) = \kappa_2 \left[\frac{2\kappa_1 + 1}{a_2\kappa_1(\mu+2)}\right]^{\kappa_2} b^{\kappa_2 - 1}$$

Bidder 1's *ex-ante* expected price as a buyer is

$$\begin{aligned} P_1^B(\pi_1^1, \pi_2^1) &= \int_0^{\sup b} bF_2 \circ \pi_2^1(b) F_1 \circ \pi_1^1(\mathrm{d}b) \\ &= \frac{a_1\kappa_1\kappa_2(\mu+2)}{(2\kappa_2+1)(\kappa_1+\kappa_2+1)} \end{aligned}$$

Using symmetry, we have

$$\begin{aligned} P_2^B(\pi_1^1, \pi_2^1) &= \frac{a_2\kappa_1\kappa_2(\mu+2)}{(2\kappa_1+1)(\kappa_1+\kappa_2+1)} \\ &= \frac{a_1\kappa_2^2(\mu+2)}{(2\kappa_2+1)(\kappa_1+\kappa_2+1)} \end{aligned}$$

Comparing $P_1^B(\pi_1^1, \pi_2^1)$ and $P_2^B(\pi_1^1, \pi_2^1)$, we get $P_1^B(\pi_1^1, \pi_2^1) > P_2^B(\pi_1^1, \pi_2^1)$.

Bidder 1's *ex-ante* expected price as a seller is

$$P_1^S(\pi_1^1, \pi_2^1) = \int_0^{\sigma_1^1(a_1)} \left[\int_{\pi_2^1(b)}^{a_2} \sigma_2^1(t_2) F_2(\mathrm{d}t_2)\right] F_1 \circ \pi_1^1(\mathrm{d}b)$$

This implies

$$\begin{aligned} \int_{\pi_2^1(b)}^{a_2} \sigma_2^1(t_2) F_2(\mathrm{d}t_2) &= \frac{(\mu+2)\kappa_1\kappa_2}{a_2^{\kappa_2}(2\kappa_1+1)} \int_{\pi_2^1(b)}^{a_2} t_2^{\kappa_2} \mathrm{d}t_2 \\ &= \frac{(\mu+2)\kappa_1\kappa_2 a_2}{(1+\kappa_2)(2\kappa_1+1)} - \frac{(\mu+2)\kappa_1\kappa_2\pi_2(b)^{\kappa_2+1}}{a_2^{\kappa_2}(1+\kappa_2)(2\kappa_1+1)} \end{aligned}$$

Using it in the expression of $P_1^S(\pi_1^1, \pi_2^1)$, we have

$$\begin{aligned} P_1^S(\pi_1^1, \pi_2^1) &= \frac{\kappa_1\kappa_2 a_2(\mu+2)}{(2\kappa_1+1)(\kappa_1+\kappa_2+1)} \\ &= \frac{\kappa_2^2 a_1(\mu+2)}{(2\kappa_2+1)(\kappa_1+\kappa_2+1)} \\ &= P_2^B(\pi_1^1, \pi_2^1) \end{aligned}$$

Using symmetry, it can be shown that

$$P_1^B(\pi_1^1, \pi_2^1) = P_2^S(\pi_1^1, \pi_2^1) = \frac{\kappa_1 \kappa_2 a_1(\mu + 2)}{(2\kappa_2 + 1)(\kappa_1 + \kappa_2 + 1)}$$

We now show $C_1(\pi_1^1, \pi_2^1) > C_2(\pi_1^1, \pi_2^1)$. To do so, let $\Lambda_i : T_j \to T_i$ be defined as $\Lambda_i(t_j) := \pi_i^1 \circ \sigma_j^1(t_j)$ for every $i, j \in N$. Then, from the expressions of bid functions and inverse bid functions, we have

$$\Lambda_1(t_2) = \frac{a_1}{a_2} t_2, \quad \Lambda_2(t_1) = \frac{a_2}{a_1} t_1$$

and

$$F_1 \circ \Lambda_1(t_2) = \left(\frac{t_2}{a_2}\right)^{\kappa_1}, \quad F_2 \circ \Lambda_2(t_1) = \left(\frac{t_1}{a_1}\right)^{\kappa_2}$$

for every $t_1 \in T_1$ and every $t_2 \in T_2$. The *ex-ante* expected belief of bidder 1 being a buyer is

$$\begin{aligned} C_1(\pi_1^1, \pi_2^1) &= \int_0^{a_1} F_2 \circ \Lambda_2(t_1) F_1(\mathrm{d}t_1) \\ &= \frac{\kappa_1}{\kappa_1 + \kappa_2} \end{aligned}$$

Symmetrically, we have

$$C_2(\pi_1^1, \pi_2^1) = \frac{\kappa_2}{\kappa_1 + \kappa_2}$$

Comparing $C_1(\pi_1^1, \pi_2^1)$ and $C_2(\pi_1^1, \pi_2^1)$, we have $C_1(\pi_1^1, \pi_2^1) > C_2(\pi_1^1, \pi_2^1)$. ■

Proof of Theorem 3. From (5) and (8), we have the inverse bid functions as

$$\phi_1^1(b) = \frac{1 + \kappa_2}{\mu \kappa_2} b, \quad \phi_2^1(b) = \frac{b}{2}, \tag{9}$$

the bid functions as

$$\theta_1^1(t_1) = \frac{\mu \kappa_2}{1 + \kappa_2} t_1, \quad \theta_2^1(t_2) = 2t_2, \tag{10}$$

the bid distributions as

$$F_1 \circ \phi_1^1(b) = \left(\frac{1 + \kappa_2}{\mu a_1 \kappa_2}\right)^{\kappa_1} b^{\kappa_1}, \quad F_2 \circ \phi_2^1(b) = \left(\frac{b}{2a_2}\right)^{\kappa_2} \tag{11}$$

and the bid densities as

$$\mathrm{D}F_1 \circ \phi_1^1(b) = \kappa_1 \left(\frac{1+\kappa_2}{\mu a_1 \kappa_2} \right)^{\kappa_1} b^{\kappa_1 - 1}, \quad \mathrm{D}F_2 \circ \phi_2^1(b) = \kappa_2 \left(\frac{1}{2a_2} \right)^{\kappa_2} b^{\kappa_2 - 1} \tag{12}$$

The supremum of bid is

$$\sup b = \theta_1^1(a_1) = \frac{\mu a_1 \kappa_2}{1+\kappa_2} = 2a_2 = \theta_2^1(a_2)$$

Let $\Omega_i : T_j \to T_i$ be defined as $\Omega_i(t_j) := \phi_i^1 \circ \theta_j^1(t_j)$ for every $t_j \in T_j$ and for every $i, j \in N$. Given $t_j \in T_j$, $\Omega_i(t_j)$ is the bid of bidder i so that it equalizes bidder j's bid $\theta_j(t_j)$. So, we have

$$\Omega_1(t_2) = \phi_1 \circ \theta_2(t_2) = \phi_1(2t_2) = \frac{2(1+\kappa_2)}{\mu \kappa_2} t_2 = \frac{a_1}{a_2} t_2$$

and

$$\Omega_2(t_1) = \phi_2 \circ \theta_1(t_1) = \phi_2 \left(\frac{\mu \kappa_2}{1+\kappa_2} t_1 \right) = \frac{\mu \kappa_2}{2(1+\kappa_2)} t_1 = \frac{a_2}{a_1} t_1$$

Given $t_1 \in T_1$ and $t_2 \in T_2$, we have

$$F_2 \circ \Omega_2(t_1) = \left(\frac{t_1}{a_1} \right)^{\kappa_2}, \quad F_1 \circ \Omega_1(t_2) = \left(\frac{t_2}{a_2} \right)^{\kappa_1} \tag{13}$$

The buyer's *ex-ante* expected payment is

$$\begin{aligned} P_1(\phi_1^1, \phi_2^1) &= \int_0^{\sup b} b F_2 \circ \phi_2^1(b) F_1 \circ \phi_1^1(\mathrm{d}b) \\ &= \frac{2a_2 \kappa_1}{\kappa_1 + \kappa_2 + 1} \end{aligned}$$

The seller's *ex-ante* expected payment received is

$$P_2(\phi_1^1, \phi_2^1) = \int_0^{\sup b} \int_{\phi_1^1(b)}^{a_1} \theta_1^1(t_1) F_1(\mathrm{d}t_1) F_2 \circ \phi_2^1(\mathrm{d}b)$$

Using (10), we have

$$\begin{aligned} \int_{\phi_1^1(b)}^{a_1} \theta_1^1(t_1) F_1(\mathrm{d}t_1) &= \int_{\phi_1^1(b)}^{a_1} \theta_1^1(t_1) f_1(t_1)(\mathrm{d}t_1) \\ &= \frac{2a_2 \kappa_1}{1+\kappa_1} - \frac{2a_2 \kappa_1}{a_1^{\kappa_1+1}(1+\kappa_1)} \phi_1(b)^{\kappa_1+1} \end{aligned}$$

Using it in the expression of $P_2(\phi_1, \phi_2)$, we have

$$P_2(\phi_1^1, \phi_2^1) = \frac{2a_2\kappa_1}{\kappa_1 + \kappa_2 + 1}$$

Comparing $P_1(\phi_1^1, \phi_2^1)$ and $P_2(\phi_1^1, \phi_2^1)$, we arrive at $P_1(\phi_1^1, \phi_2^1) = P_2(\phi_1^1, \phi_2^1)$.

We now show that $C_1(\phi_1^1, \phi_2^1) > C_2(\phi_1^1, \phi_2^1)$. From (13), we have

$$\begin{aligned} C_1(\phi_1^1, \phi_2^1) &= \int_0^{a_1} F_2 \circ \Omega_2(t_1) F_1(\mathrm{d}t_1) \\ &= \frac{\kappa_1}{\kappa_1 + \kappa_2} \end{aligned}$$

and

$$\begin{aligned} C_2(\phi_1^1, \phi_2^1) &= \int_0^{a_2} [1 - F_1 \circ \Omega_1(t_2)] F_2(\mathrm{d}t_2) \\ &= \frac{\kappa_1}{\kappa_1 + \kappa_2} \end{aligned}$$

Comparing $C_1(\phi_1^1, \phi_2^1)$ and $C_2(\phi_1^1, \phi_2^1)$, we have $C_1(\phi_1^1, \phi_2^1) = C_2(\phi_1^1, \phi_2^1)$. ■

Proof of Theorem 4. From Theorem 2, we have

$$\sigma^1(t) = \frac{\kappa(\mu + 2)}{2\kappa + 1} t, \quad P^B(\pi^1) = P^S(\pi^1) = \frac{2a\kappa}{2\kappa + 1}$$

and from Theorem 3, we have

$$\theta_1^1(t) = \frac{\mu\kappa}{1 + \kappa} t, \quad P_1(\phi^1) = P_2(\phi^1) = \frac{2a\kappa}{2\kappa + 1}$$

Comparing the above expressions, we get (1) and (2).

From Theorems 1-3 and (3)-(4), we have

$$\begin{aligned} V(\pi^1) &= \frac{\kappa(\mu\kappa + \mu + 1)}{(\kappa + 1)(2\kappa + 1)}, \quad V_1(\phi^1) = \frac{\kappa(\mu - 2)}{2\kappa + 1} + \frac{\kappa}{\kappa + 1}, \\ V_2(\phi^1) &= \frac{\kappa(2\kappa + 3)}{(2\kappa + 1)(\kappa + 1)}, \quad V^* = \frac{\kappa}{\kappa + 1} \end{aligned}$$

Comparing them, we get (3). ■

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