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Spheres of Exchange: Restricted Convertibility as Inequality-Aware Market Design*

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Abstract

Spheres of exchange (SOEs) are institutional arrangements in which goods trade freely within distinct spheres but face formal restrictions on cross-sphere exchange. Most commonly, this restricted convertibility separates subsistence goods from luxury goods. While SOEs are extensively documented by historians and ethnographers, their welfare properties remain largely unexamined in formal economic models. This paper develops a general equilibrium framework showing that SOEs can be understood as a form of inequality-aware market design. By restricting the convertibility of luxury wealth into essential goods, SOEs improve access to subsistence goods for poorer agents. We show when SOEs dominate commodity taxation and quantity rationing on utilitarian grounds, and that combining SOEs with commodity taxation can yield higher welfare than either instrument alone.

Keywords: Spheres of Exchange, Inequality-Aware Market Design, Economic History, Taxation, Rationing, Distribution and Welfare.

JEL Codes: D3, D6, H2.

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1 Introduction

The economic systems of ancient and tribal societies have long intrigued historians and economists for their complexity, adaptability, and institutional diversity (Polanyi, 1944; Keynes, 1982; Barjamovic et al., 2019; Moscona, Nunn, and Robinson, 2020; Lowes and Nunn, 2024; Boehm and Chaney, 2024). Among their most distinctive yet understudied institutions are *spheres of exchange* (SOEs)—systems in which goods trade freely within distinct domains but face restrictions on trade across them, within a closed economy. Spheres of exchange have been documented across a wide range of contexts, including Melanesian societies (Malinowski, 1922), African economies (P. Bohannan, 1959; Barth, 1967), and ancient Mediterranean and Near Eastern economies (Keynes, 1982; Powell, 1996). Despite their prevalence, such exchange restrictions remain largely absent from formal economic theory, which typically assumes full convertibility across goods.

In this paper, we propose that spheres of exchange constitute an overlooked form of inequality-aware market design. Rather than redistributing through ex-post transfers or rationing, these institutions embed distributive objectives directly into the rules governing exchange by restricting cross-domain convertibility. This perspective aligns with a growing literature showing that market design and mechanism design can incorporate distributional objectives directly into allocation rules (Condorelli, 2013; Scheuer and Wolitzky, 2016; Dworczak, Kominers, and Akbarpour, 2020; Akbarpour, Dworczak, and Kominers, 2024). We show that when inequality concerns center on access to essential goods, such design constraints can improve welfare and, under clearly defined conditions, dominate conventional redistributive instruments.

We formalize this logic in a general equilibrium exchange economy with two classes of goods: essentials, whose equitable distribution is directly welfare-relevant, and luxuries, whose trade generates surplus but whose distribution is not itself a direct social objective. To isolate the redistributive mechanism, we model an exchange economy where all agents are identically endowed with tradable leisure, abstracting from production and labor supply. Agents differ, however, in their remaining endowments: some hold luxury goods such as silver, others possess essential commodities like barley, while the poorest agents are endowed only with leisure. In a two-sphere regime, essentials and leisure trade in one market and luxuries in another, with no exchange permitted across spheres. In a laissez-faire equilibrium,[1] wealthier agents can convert luxury endowments into essentials, bidding up prices and crowding out access for poorer agents. By restricting this convertibility, spheres of exchange prevent such cross-sphere substitution. As a result, demand from wealthier agents

[1] The allocation arising from a free-market Walrasian equilibrium.

is curtailed, the price of essentials falls relative to leisure, and utilitarian welfare improves relative to laissez-faire.

Having established that SOEs can improve upon laissez-faire, we now turn to a central institutional question: how do they compare to alternative redistributive mechanisms? We address this by formally comparing SOEs to two canonical instruments: quantity rationing (QR), which directly caps consumption of essential goods (Tobin, 1970; Weitzman, 1977), and commodity taxation (CT), which raises revenue through taxes on luxury trade that can be used for redistribution (Ramsey, 1927; P. A. Diamond and James A. Mirrlees, 1971; P. Diamond and Mirlees, 1971). We maintain the classical public finance constraint that the planner cannot make lump-sum transfers.[2] Our analysis shows that, under clearly defined conditions, SOEs can dominate both instruments on utilitarian grounds.

We first show that spheres of exchange dominate quantity rationing when agents have heterogeneous preferences over essential goods. Consider two substitutable staples such as barley and wheat. Rationing imposes good-specific caps, forcing agents toward a common bundle regardless of taste. In an unequal economy, relaxing these caps to accommodate preference differences merely amplifies demand from the wealthy, raising prices and diluting redistribution. In contrast, spheres of exchange allow unrestricted trade within the essential sphere, so prices allocate goods efficiently across preference types while keeping the equilibrium price low. This price-mediated allocation within the sphere delivers a welfare advantage over rationing.

The comparison with commodity taxation reveals a trade-off between revenue and deadweight loss. Taxation generates revenue and indirectly curbs demand for essential goods, but it distorts trade within the luxury sphere. Spheres of exchange generate no revenue, but they restrict the cross-sphere substitution of luxuries for essentials while preserving efficiency within each sphere. The welfare ranking turns on the magnitude of the surplus from luxury trade. When this surplus is small, high taxation can restrict demand and raise revenue with modest welfare-loss; when it is very large, even low taxes yield ample revenue with little efficiency loss. At intermediate surplus levels, however, spheres of exchange combine effective demand restriction with no deadweight loss in luxury trade—and thus dominate commodity taxation on utilitarian grounds.

While our baseline analysis considers fully disjoint spheres, historical evidence points to more nuanced arrangements in which limited cross-sphere exchange was permitted at regulated rates. Such *partial spheres of exchange* appear in Mesopotamia's dual monetary system (Powell, 1996; Cripps, 2017) and in African societies where cross-sphere transactions were allowed, subject to social costs (P. Bohannan, 1959). We extend the model to incor-

[2]This precludes the second welfare theorem.

porate regulated convertibility between spheres, characterize the optimal conversion rate, and show that partial spheres of exchange and commodity taxation can be complementary instruments, jointly yielding higher welfare than either instrument alone under appropriate conditions.

Implementing spheres of exchange does not require richer aggregate information than conventional taxation or rationing. The central challenge is enforcement—preventing cross-sphere arbitrage. As we discuss in Section 4, historical societies sustained spheres of exchange through social norms in small-scale settings and state monitoring in larger polities. These cases illustrate that restricted convertibility is institutionally feasible and suggest how its underlying logic may inform the design of modern redistributive mechanisms, including in-kind transfers and rationing systems.

In sum, this paper shows that spheres of exchange can be understood as a form of inequality-aware market design that restricts the convertibility of wealth across goods rather than prices or quantities. In a general equilibrium setting, such restrictions improve access to essential goods, accommodate heterogeneous preferences, and can dominate quantity rationing and commodity taxation under well-defined conditions on utilitarian grounds. By formalizing a central institution from economic anthropology within modern economic theory, the paper identifies restricted convertibility as a distinct and welfare-relevant redistributive principle.

1.1 Related Literature

Our work bridges ethnographic studies on spheres of exchange in tribal and ancient societies across Africa, Asia, and the Pacific (Malinowski, 1922; P. Bohannan, 1959; P. Bohannan and Dalton, 1965; Barth, 1967; Powell, 1996; Sillitoe, 2006) with economic theory by formalizing the leading interpretation on why such systems emerged within a general-equilibrium framework. While spheres of exchange are well documented in anthropology, they have received little formal treatment in economics. To the best of our knowledge, no existing economic model has analyzed spheres of exchange as a general-equilibrium mechanism for redistributing access to essential goods.

A modern institutional parallel can be found in kidney-exchange programs, where legal prohibitions on monetary transactions create de facto separate spheres of exchange (cash vs. organs) (A. Roth, Sönmez, and Unver, 2004; Becker and Julio Jorge Elías, 2007; Julio J. Elías, Lacetera, and Macis, 2019; Akbarpour, Li, and Gharan, 2020; Akbarpour, Combe, et al., 2024). While that literature focuses on designing efficient matching mechanisms within a constrained sphere, we analyze how the intentional creation of such spheres can itself be a redistributive instrument for essential commodities.

Conceptually, our work connects to the emerging literature on *inequality-aware market design*, which studies how market rules can be structured to internalize distributional objectives (Bulow and Klemperer, 2012; Condorelli, 2013; Dworczak, Kominers, and Akbarpour, 2020; Akbarpour, Dworczak, and Kominers, 2024). This strand shows that restricting choice sets or modifying allocation mechanisms can improve welfare when fiscal redistribution is limited or infeasible. We extend this logic by providing a general-equilibrium foundation for a historical market-design institution: spheres of exchange that achieve redistribution without taxation or transfers.

Our paper also contributes to the canonical public-finance literature on redistribution (Ramsey, 1927; Tobin, 1970; P. Diamond and Mirlees, 1971; P. A. Diamond and James A. Mirrlees, 1971; J. A. Mirrlees, 1971; Atkinson and Stiglitz, 1976; Weitzman, 1977; Naito, 1999; Saez, 2002; Saez, 2004; Scheuer and Ivan Werning, 2017; Scheuer and Slemrod, 2019; Gadenne, 2020; Hummel and Ziesemer, 2023). Building on results showing that Atkinson and Stiglitz (1976)'s theorem on uniform commodity taxation breaks down when marginal costs are upward sloping (Naito, 1999; Hummel and Ziesemer, 2023), we characterize how SOEs achieve redistribution by restricting demand to lower the relative price of essential goods. More centrally, we derive sufficient conditions under which this restriction yields higher utilitarian welfare than either commodity taxation or quantity rationing.[3]

Finally, our analysis of partial spheres—where limited cross-sphere exchange is permitted at a regulated rate—connects to the literature on commodity money and dual monetary systems (Barro, 1979; Sargent and Wallace, 1983; Sargent, 2019; Velde and Weber, 2000). While these studies suggest that dual-currency regimes typically result in lower welfare than single-currency systems (Velde and Weber, 2000; Sargent, 2019), dual monetary arrangements can, in fact, enhance welfare when redistribution is an explicit policy objective.

The rest of the paper is structured as follows. Section 2 provides the historical and ethnographic background on spheres of exchange. Section 3 develops the model and analyzes the welfare properties of SOEs compared to laissez-faire, quantity rationing, and commodity taxation. Section 4 discusses implementation, and Section 5 concludes.

2 Background

Anthropologists and historians have long documented the existence of *spheres of exchange* (SOEs)—systems where goods circulated within distinct economic and moral domains. One

[3]Although our model abstracts from non-linear labor taxation, Scheuer and Iván Werning (2016) shows that linear tax models can nest non-linear labor taxation.

of the earliest observers of this phenomenon, Keynes (1982), described how early Greek economies operated with multiple "standards of value"—grain for subsistence, livestock for wealth storage, and metals for trade—each functioning within its own exchange sphere.

Beyond historical cases, ethnographers have documented SOEs in tribal societies. In the Trobriand Islands, the ceremonial *kula* exchange network involved prestige goods circulating in a closed circuit, separate from everyday subsistence markets (Malinowski, 1922). The most influential ethnographic study of SOEs comes from Laura and Paul Bohannan's analysis of the Tiv people in Nigeria (L. Bohannan and P. Bohannan, 1953; P. Bohannan, 1959). The Tiv maintained three distinct exchange spheres: (i) a subsistence sphere for food, tools, and small livestock; (ii) a wealth sphere for high-value goods such as brass rods, cattle, and cloth; and (iii) a marriage sphere involving the exchange of rights in women. Movement between spheres was tightly restricted by social norms.

Subsequent anthropological research documents similar exchange restrictions across Africa and the Pacific, including among the Lele, Ndembu, Ganda, Tallensi, Fur, Maori, and Orokaiva (P. Bohannan and Dalton, 1965; Barth, 1967; Firth, 1929; Hogbin, 1951). Across these contexts, SOEs have been interpreted as limiting elite accumulation and preserving access to subsistence goods.

Synthesizing these findings, Sillitoe (2006) argues that most systems can be understood as variations of a two-sphere structure separating subsistence from wealth or prestige goods. While more complex societies featured additional spheres, this binary organization captures the essential redistributive logic: by preventing the conversion of wealth into essential goods, societies are argued to have curtailed resource concentration and promoted egalitarian access to livelihood necessities. Guided by this insight, our theoretical framework models a two-sphere system that captures the key redistributive mechanism underlying these ethnographic examples.

We now turn to ancient Mesopotamia, a case that exemplifies a more nuanced system which we term *partial spheres of exchange* (PSOEs). Here, a dual monetary system—using barley for subsistence and silver for luxury trade—allowed for limited cross-sphere exchange at a state-regulated rate. Similar principles of regulated conversion also appear in the Tiv system described earlier.

2.1 Partial Spheres of Exchange: A Tale of Two Monies

Mesopotamia, often called the "cradle of civilization," flourished between 3100 and 539 BCE across present-day Iraq, Syria, and Iran. Despite advances in writing, law, and administration, it remained highly hierarchical, with economic power concentrated among temples, palaces, and a small elite, while most laborers and tenants relied on these institutions for subsistence (Mieroop, 2016; Postgate, 1992). A small class of merchants engaged in long-distance trade, linking Mesopotamia to regions as far as Anatolia and the Indus Valley, but their activities primarily served elite demand.

The role of markets in Mesopotamia has long divided scholars. Substantivists, following Polanyi (1944), view the economy as embedded in social and political institutions, while formalists highlight price formation, private ownership, and market exchange (Snell, 1982; Silver, 1983; Powell, 1999). Evidence from cuneiform records suggests that both perspectives hold: prices for commodities such as barley fluctuated with supply, yet the state routinely intervened to stabilize access to subsistence through rations, wage regulation, and debt cancellation (M. T. Roth, 1997; Renger, 1994; Hudson, 2018). This blend of market activity and redistribution formed the institutional foundation for Mesopotamia's distinctive monetary system.

Archaeological and textual evidence shows that both silver and barley functioned as money—acting as a medium of exchange, units of account, and standards for deferred payment (Powell, 1996; Hudson, 2004; Cripps, 2017). Barley, used for everyday transactions and wages, acted as "cheap money" for essentials, while silver facilitated high-value and long-distance trade. Although formalists and substantivists differ on the scope of market exchange, both agree that the state fixed the exchange rate between the two: one *shekel* of silver ($\approx 8.3g$) equaled one *gur* of barley ($\approx 300kg$) (Cripps, 2017). This fiat-determined rate effectively established two monetary spheres—one for subsistence and one for luxury exchange—linked through a regulated conversion rule.

By setting the barley–silver rate above its market value, the state could indirectly tax silver holders, redistributing purchasing power toward those transacting mainly in barley. The system thus combined market pricing within each sphere with state control across spheres, enabling redistribution without direct transfers. Our framework formalizes this insight by modeling how such a dual-currency regime can implement a form of *partial spheres of exchange* (PSOE), balancing efficiency in trade with equitable access to essentials.

Comparable features appear in ethnographic accounts of the Tiv, where cross-sphere conversion was normally forbidden but permitted under exceptional circumstances at social cost (P. Bohannan, 1959). These controlled exceptions illustrate how limited convertibility can preserve subsistence access while maintaining efficiency. In the next section, we formalize this logic by modeling how SOEs and PSOEs sustain redistribution in general equilibrium.

3 Model

We consider a pure exchange economy with three goods—foodgrains (f), leisure (l), and silver (s)—and three types of agents: landowners (A), landless laborers (B), and traders (C).[4] We normalize the mass of landowners to 1; b and c denote the masses of laborers and traders.[5]

Endowments. The endowment structure is summarized in Table 1. Landowners are endowed with one unit of foodgrain and L units of leisure; traders are endowed with L units of leisure and S units of silver and landless laborers possess L units of leisure only.[6]

Type	**Foodgrain**	**Leisure**	**Silver**
Landowner	1	L	0
Landless Laborer	0	L	0
Trader	0	L	S

Table 1: Endowment Distribution Across Agent Types

Preferences. All agents share quasi-linear preferences over foodgrains, silver, and leisure:

$$U(f, s, l) = k \ln f + s + l,$$

where $k > 0$ governs the intensity of preference for foodgrains. Foodgrains are *essential*—the marginal utility of consumption approaches infinity as $f \to 0$—while silver and leisure are non-essential, each yielding constant marginal utility even at zero consumption.[7]

Under these preferences, the individual demand for foodgrains depends only on its price p_f and income m:

$$f(p_f, m) = \begin{cases} \frac{k}{p_f} & \text{if} \quad m \geq k \\ \frac{m}{p_f} & \text{if} \quad m < k \end{cases}.$$

[4] We use the term landowners and landlords interchangeably. We also use the term rich for traders. Landless laborers are also called the poor or simply laborers.

[5] Only relative population proportions matter for equilibrium outcomes.

[6] Leisure in this model is simply a tradable non-essential good with constant marginal utility. It is not labor supply and does not enter any production technology. Endowing all agents with L units ensures that each has a non-zero endowment that can be traded for essentials even when silver holdings are zero.

[7] Constant marginal utility at zero defines non-essential goods in our framework. We later show that the results extend to a broader class of quasi-linear preferences.

Laissez-Faire (LF) Equilibrium. A *laissez-faire* equilibrium is a vector of prices (p_f^{LF}, p_s, p_l) and allocations $\{f_i, s_i, l_i\}_{i\in\{A,B,C\}}$ such that: (i) each agent maximizes utility subject to their budget constraint, and (ii) markets for foodgrains, leisure, and silver clear.

We normalize the price of leisure to one, $p_l = 1$. From our specification of preferences, this implies $p_s = 1$. Equilibrium nominal incomes for the three types are therefore:

$$m_A = p_f + L, \qquad m_B = L, \qquad m_C = S + L.$$

We assume $L < k$, and that the silver endowment S is sufficiently large to ensure that in equilibrium, both landowners and traders have sufficient income ($m_A = p_f + L > k$ and $m_C = S + L > k$) to demand foodgrains at their satiation level, leaving only laborers income-constrained.

Substituting the demand functions and imposing market-clearing conditions yields the equilibrium price of foodgrains:

$$p_f^{LF} = bL + (1 + c)k.$$

Full allocations under the *laissez-faire* regime are summarized in Table 2.

Definition 1 (Spheres of Exchange (SOE) Economy)**.** A *Spheres of Exchange (SOE)* economy consists of two segmented spheres of trade:

1. The *essential-goods sphere*, in which foodgrains (f) and leisure (l) may be exchanged;

2. The *luxury-goods sphere*, in which only luxuries (s) are exchanged.

No trade occurs between the two spheres: luxuries cannot be exchanged for foodgrains or leisure, and vice versa.

Individual Optimization Problem. Let $(\hat{f}, \hat{L}, \hat{S})$ denote an agent's initial endowments of foodgrains, leisure, and silver, respectively. Given the price vector (p_f, p_l, p_s), the agent chooses (f, l, s) to maximize

$$\max_{f,l,s} U(f, s, l),$$

subject to the sphere-specific budget constraints:

$$p_f(f - \hat{f}) + p_l(l - \hat{L}) \leq 0, \qquad \text{(essential-goods sphere)},$$
$$p_s(s - \hat{S}) \leq 0, \qquad \text{(luxury-goods sphere)}.$$

Solving this yields individual's foodgrains demands:

$$f^{B,C}(p_f) = \frac{L}{p_f}, \qquad f^A(p_f) = \frac{k}{p_f}.$$

Landowners can sell their foodgrain endowment within the essential sphere, so they are unconstrained and consume k/p_f. Traders are income-constrained in the essential sphere because their silver cannot be used to purchase food; they can spend only their leisure endowment L. In SOE, both the traders and laborers are constrained to spend their limited leisure endowment in the essential sphere, while landowners, as net sellers of food, have sufficient income within that sphere to reach satiation.

Equilibrium under SOE. An equilibrium under *spheres of exchange* (SOE) is a vector of prices (p_f, p_l, p_s) and allocations $\{f_i, s_i, l_i\}_{i\in\{A,B,C\}}$ such that: (i) each agent maximizes utility subject to the sphere-specific budget constraints; and (ii) markets for foodgrains, leisure, and silver clear. We normalize $p_l = 1$.

Proposition 1. *Under the maintained assumption $L < k$, in the SOE economy, the equilibrium foodgrain price is*

$$p_f^{SOE} = (b + c)L + k,$$

and the corresponding foodgrain consumption allocations are given in Table 2.

Type	**Endowments**	**Foodgrains (LF)**	**Foodgrains (SOE)**
Landowner (A)	(1, L, 0)	$\frac{k}{bL+(1+c)k}$	$\frac{k}{(b+c)L+k}$
Landless (B)	(0, L, 0)	$\frac{L}{bL+(1+c)k}$	$\frac{L}{(b+c)L+k}$
Trader (C)	(0, L, S)	$\frac{k}{bL+(1+c)k}$	$\frac{L}{(b+c)L+k}$

Table 2: Endowments and foodgrain allocations under Laissez-Faire (LF) and Spheres of Exchange (SOE). Prices: $p_l = 1$, $p_s = 1$ (from quasi-linearity), $p_f^{LF} = bL + (1+c)k$, and $p_f^{SOE} = (b+c)L + k$.

Relative to laissez-faire, the SOE equilibrium lowers the price of essentials by restricting traders' demand, improving the consumption share of the poor.

Proposition 2. *Fix $L, k, S > 0$ such that $L < k$ and $S + k > L$. There is some number $\hat{B} = B(L, k, S, c)$ such that for all $b > \hat{B}$, utilitarian welfare $W = U^A + bU^B + cU^C$ is higher* *under spheres of exchange (SOE) than under laissez-faire (LF).*

$$W^{SOE} > W^{LF}.$$

The intuition behind Proposition 2 is straightforward. By restricting trader demand, the SOE regime lowers the equilibrium price of essentials and reallocates foodgrains toward laborers. Because utility from foodgrains is concave, this redistribution improves overall welfare whenever the laboring class constitutes a sufficiently large share of the population. Formally, the result follows from Appendix A.1. Numerical simulations in Figure 1 illustrate that the qualitative mechanism — restricted convertibility lowers essential prices — does not rely on the asymptotics.[8]

Comparison with Alternative Policies. We now compare the spheres of exchange (SOE) regime with two conventional redistributive instruments: (i) *quantity rationing* (QR), which caps individual foodgrain purchases,[9] and (ii) *commodity taxation* (CT), which imposes a linear tax on selling luxury good(s).

Remark 1 (Equivalence in the Baseline Model)*.* With three goods, the SOE allocation can be replicated by a suitable QR or CT policy. Prohibitively high commodity taxes sever the link between spheres, while rationing can mandate the SOE consumption levels directly.

This equivalence, however, is fragile. It depends crucially on the knife-edge assumption that only one essential and one luxury exist.

(i) **Multiple Essentials:** With heterogeneous preferences over essentials (e.g., barley vs. wheat), SOEs can dominate QR. Rationing imposes good specific caps that prevent efficient allocation of expenditure, while SOEs allow unrestricted trade *within* the essential sphere, harnessing prices to allocate goods efficiently.

(ii) **Multiple Luxuries:** With gains from trade among luxuries, SOEs can dominate CT. Commodity taxation distorts luxury-luxury trade, creating deadweight loss, while SOEs restrict cross-sphere conversion *without* distorting intra-sphere efficiency.

SOEs uniquely restrict only cross-sphere conversion while leaving the within-sphere allocation governed by undistorted market prices.

[8] Asymptotic argument is used to make the algebra of the welfare comparison transparent.

[9] QR can be represented equivalently as an infinite marginal tax beyond the cap (Gadenne, 2020).

3.1 Heterogeneity in the Essential-Goods Sphere: Case Against *QR*

We now extend the model to incorporate heterogeneity in preferences over foodgrains. The economy contains four goods: two foodgrains, barley (f_1) and wheat (f_2), leisure (l), and silver (s). Agents differ in their taste for foodgrains. Preference types $t \in \{1, 2\}$ are given by utilities:

$$U_1(f_1, f_2, l, s) = \alpha k \ln f_1 + (1 - \alpha) k \ln f_2 + l + s,$$
$$U_2(f_1, f_2, l, s) = (1 - \alpha) k \ln f_1 + \alpha k \ln f_2 + l + s,$$

where $\alpha \in [0, 1]$ governs the relative preference for barley.[10]

Assumption 1. Preference types (U_1, U_2) occur with equal probability in the population and are independent of endowment types.[11]

Endowments. The endowment structure is shown in Table 3. **Landowners** are endowed with 1/2 unit of each foodgrain implying aggregate endowments of barley and wheat are equal. **Landless Laborers** are endowed only with leisure. **Traders** are endowed with leisure and silver.

Spheres of Exchange

Under the SOE regime, barley (f_1), wheat (f_2), and leisure (l) are traded within the essential sphere, while silver (s) trades exclusively in the luxury sphere. No cross-sphere trade is permitted. A key feature is that trade remains unrestricted *within* the essential sphere.

Agent's Problem. Given endowments $(\hat{f}_1, \hat{f}_2, \hat{L}, \hat{S})$ and prices $(p_{f_1}, p_{f_2}, p_l, p_s)$, a type-$t$ agent solves

$$\max_{f_1, f_2, l, s} U_t(f_1, f_2, l, s)$$

[10] For interpretation, it is useful to focus on the case $\alpha \neq \frac{1}{2}$, since $\alpha = \frac{1}{2}$ collapses the two preference types into a single representative type.

[11] This assumption delivers a symmetric benchmark that isolates the key inefficiency of QR—good-specific caps—rather than effects driven by preference imbalance. Without symmetry, rationing would interact with heterogeneous tastes in more complex ways, obscuring the mechanism we wish to highlight.

subject to

$$p_{f_1}(f_1 - \hat{f}_1) + p_{f_2}(f_2 - \hat{f}_2) + p_l(l - \hat{L}) \leq 0, \qquad \text{(essential sphere)},$$
$$p_s(s - \hat{S}) \leq 0, \qquad \text{(luxury sphere)}.$$

Equilibrium. Since the probability of each preference type is equal and the aggregate endowments of barley and wheat are equal, the SOE equilibrium is symmetric and satisfies $p_{f_1} = p_{f_2} = p$. The corresponding allocations appear in Table 4.

Quantity Rationing

Under quantity rationing (QR), the planner imposes caps $q^{QR} = (q_1^{QR}, q_2^{QR})$ on net purchases of each foodgrain. A type-t agent therefore solves

$$\max_{f_1, f_2, l, s} U_t(f_1, f_2, l, s)$$

subject to the (laissez-faire) budget constraints and the rationing constraint

$$f_i - \hat{f}_i \ \leq \ q_i^{QR}, \qquad i \in \{1, 2\}.$$

A QR equilibrium consists of a cap vector q^{QR} (chosen exogenously by the planner), a price vector p^{QR}, and associated allocations such that: (i) each agent maximizes utility subject to budget and rationing constraints, and (ii) markets clear in both spheres.

Remark 2. Under our primitives, the QR regime admits a continuum of equilibria—one for each admissible rationing vector q^{QR}.[12] In contrast, both the LF and SOE regimes yield unique competitive equilibria.

Remark 3. Under Assumption 1, any optimal QR policy must impose symmetric rationing caps: $q_{f_1}^{QR} = q_{f_2}^{QR}$. Under our primitives, any solution to the planner's problem for picking the optimal cap is convex and thus is characterized by a first-order condition. By symmetry, this FOC is the same both goods and hence the optimal cap must be identical. The resulting demand for essential goods under a symmetric QR policy is reported in Table 5.

Proposition 3 (SOE dominates QR for the poor)**.** *Suppose (i) aggregate endowments of barley and wheat coincide, and (ii) preference types occur with equal probability and are independent of endowment types. Then landless laborers attain weakly higher utility under SOE than under any QR equilibrium.*

[12] Admissible rationing vector—that is, caps that do not exceed aggregate supply—there exists a corresponding competitive equilibrium.

Intuition. With two essential goods and heterogeneous tastes, the key distinction between SOE and QR lies in how they allocate scarce purchasing power within the essential sphere. Under SOE, barley, wheat, and leisure trade freely, so prices adjust to allocate each grain efficiently across preference types. Laborers all enter the essential sphere with the same income L, and—because their marginal utilities of income coincide—this symmetric allocation is the welfare-maximizing way to distribute their total purchasing power.

QR breaks this efficiency. A rationing policy imposes good-specific caps—effectively forcing each agent to buy a fixed bundle of barley and wheat regardless of their relative tastes. Since rich agents have weakly higher essential-sphere income than poor agents under any QR policy, and essentials are normal goods, these caps cannot depress rich demand below poor demand. Aggregate feasibility then requires the QR equilibrium price to be weakly higher than the SOE price. But higher prices make every poor type consume less barley and wheat than under SOE. Thus no QR policy can raise poor consumption above SOE, and SOE weakly dominates QR for laborers.[13] Formal proof is provided in the appendix A.2.

Generalization and Robustness: SOE vs. QR

Theorem 1 (Large-b economy dominance of SOE over QR for symmetric CES utility)**.** *Fix $L > 0$, $c > 0$, $\sigma < 0$, and $\alpha \in (0, 1)$. Consider a sequence of economies indexed by the number of landless laborers $b \in \mathbb{N}$. Each economy has c traders and one landlord, holding a normalized endowment of half of each foodgrain. The endowment of the luxury for each trader is $S(b)$, where $\lim_{b\to\infty} S(b) = \infty$.*

Preferences in all these economies are given by two CES types $t \in \{1, 2\}$ over the foodgrains (f_1, f_2), leisure l, and silver s:

$$U_1(f_1, f_2, l, s) = \alpha\frac{f_1^\sigma}{\sigma} + (1-\alpha)\frac{f_2^\sigma}{\sigma} + l + s,$$
$$U_2(f_1, f_2, l, s) = (1-\alpha)\frac{f_1^\sigma}{\sigma} + \alpha\frac{f_2^\sigma}{\sigma} + l + s.$$

For each regime $R \in \{\text{SOE}, \text{QR}\}$, let utilitarian welfare be

$$W_b^R = U^{A,R} + b\,U^{B,R} + c\,U^{C,R},$$

where $U^{A,R}$, $U^{B,R}$, and $U^{C,R}$ denote, respectively, the landlord, representative poor, and representative rich utilities in regime R in the economy with b landless laborers.

[13] This difference in utilitarian welfare become strictly higher whenever $\alpha \neq \{0, 1/2, 1\}$.

Under quantity rationing (QR), the planner imposes a symmetric per-good cap $q \geq 0$ on net purchases of the essential goods by all non-landlord agents:

$$f_i - \hat{f}_i \;\leq\; q, \qquad i \in \{1, 2\},$$

and let $W_b^{\mathrm{QR}}(q)$ denote the corresponding equilibrium welfare in the economy with b landless laborers.

Then there exists $\widehat{B} = \widehat{B}(L, c, \sigma, \alpha)$ such that, for all $b \geq \widehat{B}$ and all $q \geq 0$,

$$W_b^{\mathrm{SOE}} \;\geq\; W_b^{\mathrm{QR}}(q).$$

Proof Idea. The formal proof is provided in Appendix A.2. The argument proceeds in two steps. First, we extend Proposition 3 to the CES case and show that, for the class of symmetric CES preferences above, landless laborers (the poor) attain weakly higher utility under SOE than under any QR policy. The intuition is analogous. Second, when $\sigma < 0$ and the mass of the poor is sufficiently large (i.e. for b large), the poor's welfare contribution dominates utilitarian social welfare. Since SOE is already weakly better than QR for each poor agent, it follows that aggregate utilitarian welfare under SOE is higher than under any QR policy once b exceeds a threshold $\widehat{B}(c, L, \sigma, \alpha)$. In this sense, for large economies with many poor agents, SOE dominates quantity rationing under symmetric CES preferences. We illustrate this dominance using a numerical simulation in Figure 2.

Our assumption on preference types implies that landless agents differ only in their relative tastes for the two essential goods, while sharing the same marginal utility of income in the essential sphere. For the log and symmetric CES specifications above, when all agents face the same essential-good prices and have identical essential-sphere income, their indirect utility functions coincide up to an additive constant, so their marginal utility of income is identical. This property is closely related to the assumption discussed in Eden and Freitas (2023) and Doligalski et al. (2025) that tastes do not affect the marginal utility of disposable income. In our setting, the only difference across types is the preferred mix of f_1 (barley) and f_2 (wheat); given the same total income available for essentials, an additional unit of income has the same value regardless of type.

The theorem applies when $\sigma < 0$, which implies that the price elasticity of demand for essentials is $\varepsilon = \frac{1}{\sigma - 1} \in (-1, 0)$. This restriction is economically natural: staple foodgrains exhibit inelastic demand in both historical and modern contexts.[14]

[14] For a summary study about price elasticity estimates, see Andreyeva, Long, and Brownell (2010). For a more contemporary study on own price elasticity of rice in a developing country context: see, Siddique and Salam (2020) which estimate it at -0.821. For studies of Mesopotamian grain markets see Postgate (1992)

Remark 4. As $b \to \infty$, equilibrium prices of essentials diverge under all regimes. To ensure that traders remain unconstrained in the laissez-faire economy as the economy scales, we allow their silver endowment $S(b)$ to grow without bound. This normalization isolates the welfare effects of rationing and convertibility restrictions from trivial wealth constraints.

3.2 Trade within the Luxury-Goods Sphere: The Case Against Commodity Taxes

We now compare spheres of exchange (SOE) with commodity taxation (CT). Under CT, the planner sets an ad valorem tax τ on all luxury-good transactions. The tax generates revenue for redistribution but drives a wedge between buyer and seller prices. In the simple three-good economy of Section 3, a prohibitively high tax on the *single* luxury good severs the link between luxuries and essentials, thereby replicating the SOE allocation. This apparent equivalence is, however, highly fragile. It breaks down once the luxury sphere contains multiple goods with potential gains from trade. The crucial distinction is that CT distorts *all* luxury-good transactions, while SOE restricts only *cross-sphere* conversions and leaves *within-sphere* trade undistorted.

To formalize this point, we extend the model to include two luxury goods, s_1 (silver) and s_2 (lapis lazuli). Traders differ in their relative valuation of these goods, creating scope for mutually beneficial exchange. There are two equally likely preference types $t \in \{1, 2\}$ with utilities

$$U_1(f, l, s_1, s_2) = k \log f + l + \tfrac{1}{4}s_1 + \tfrac{1}{2}s_2,$$
$$U_2(f, l, s_1, s_2) = k \log f + l + \tfrac{1}{2}s_1 + \tfrac{1}{4}s_2.$$

Type 1 agents value s_2 more highly than s_1, while type 2 agents have the opposite ranking. Both types share identical preferences over essentials and leisure. This heterogeneity creates the potential for Pareto-improving luxury–luxury exchange among traders.

The endowment structure is summarized in Table 6.

Spheres of Exchange

Under the SOE regime, traders can reallocate their luxury endowments freely among themselves but are prohibited from using luxury wealth to purchase essentials. Consequently, their effective income in the essential-goods sphere is limited to their leisure endowment, L, just as it is for landless laborers. Landowners, as the sole net suppliers of foodgrains, retain

and Hudson (2018).

their higher effective income from selling their endowment. Let p^{SOE} denote the Walrasian price of the essential good. Individual demands are

$$(f_A, f_B, f_C) = \big(\tfrac{k}{p^{SOE}},\ \tfrac{L}{p^{SOE}},\ \tfrac{L}{p^{SOE}}\big),$$

and market clearing implies

$$p^{SOE} = (b+c)L + k = p^{LF} - c(k-L).$$

Since $L < k$, SOE lowers the price of essentials relative to laissez-faire and strictly increases poor consumption. This price reduction occurs because the segregation of spheres prevents traders from bidding up the price of foodgrains with their luxury wealth. (Derivations are in Appendix A.3.)

Commodity Taxation

Under a commodity tax (CT) regime, the planner imposes an ad valorem tax τ on all luxury-good transactions. This drives a wedge between the price received by the seller, p_s, and the price paid by the buyer, $(1+\tau)p_s$. The resulting revenue is redistributed either via (i) direct lump-sum transfers (DT) or (ii) subsidies on the essential good (SB). A critical insight is that the effect of τ depends on whether it is low (non-distortionary) or high (distortionary).

Non-Distortionary Taxation ($0 \le \tau \le 1$). Given the utility coefficients $(\frac{1}{4}, \frac{1}{2})$ for s_1 and s_2, the pre-tax luxury price in equilibrium is normalized to $p_s = \frac{1}{2}$, so mutually beneficial luxury–luxury trade generates gains up to $\frac{1}{4}$ per unit. For $\tau \le 1$ the tax wedge is small enough that the net surplus from trade remains positive, hence the entire luxury endowment S is brought to market and reallocated efficiently. The equilibrium seller price remains $p_s = \frac{1}{2}$. Total tax revenue is collected from all luxury transactions and is given by:

$$R^{\text{ND}}(\tau) = c \cdot \frac{\tau}{1+\tau} \cdot p_s \cdot S = c \cdot \frac{1}{2} \cdot \frac{\tau}{1+\tau} \cdot S.$$

This revenue function is maximized at the boundary $\tau = 1$, yielding $R^{\text{ND}}(\tau = 1) = c \cdot S/4$. In this range of τ, CT raises revenue but fails to restrict traders' demand for essentials, as they can still fully monetize their luxury wealth.

Distortionary Taxation ($\tau > 1$). When $\tau > 1$, the tax wedge exceeds the potential gains from luxury trade. Consequently, all voluntary trade within the luxury sphere collapses.

Traders now only sell luxuries to raise funds for essential consumption after exhausting their labor income. Their demand for the essential good becomes:

$$f = \max\left\{\frac{2k}{p_f(1+\tau)}, \frac{L}{p_f}\right\}. \tag{1}$$

where the first term corresponds to liquidating luxuries and the second to relying solely on leisure income. Revenue is generated only when the first branch is active, i.e. when $\frac{2k}{1+\tau} > L$. Thus, the revenue function is:

$$R^{\mathrm{D}}(\tau) = c \cdot \left(\frac{2k}{1+\tau} - L\right)\tau, \qquad \tau > 1.$$

Maximizing $R^{\mathrm{D}}(\tau)$ with respect to τ yields the interior optimum:

$$\tau^* = \sqrt{\frac{2k}{L}} - 1,$$

with corresponding maximal revenue:

$$R^{\mathrm{D}}(\tau^*) = c \cdot \left(\sqrt{2k} - \sqrt{L}\right)^2.$$

A crucial result is that $R^{\mathrm{D}}(\tau^*)$ is *independent of the luxury endowment S*. This is because the revenue cap is determined by the traders' need to meet their subsistence demand k, not by the size of their luxury stock. The deadweight loss from this policy, however, is proportional to S, since $\tau > 1$ completely shuts down mutually beneficial luxury–luxury exchange. (Full derivations are in Appendix A.3.)

Direct Transfers versus Subsidies

Given a fixed amount of revenue R, the planner must choose how to redistribute it. We compare two instruments: direct lump-sum transfers to the poor (DT) and subsidies on the essential good (SB). This comparison clarifies how revenue raised through commodity taxation operates as a redistributive tool, and thereby how its effects compare to the redistribution achieved by spheres of exchange.

Direct Transfers (DT). Under DT, the revenue R^{ND} is distributed equally to the b landless laborers, so each receives R^{ND}/b. Let p^{DT} be the resulting equilibrium price. The consumption demands are:

$$f_B = \frac{L + R^{\text{ND}}/b}{p^{\text{DT}}}, \qquad f_A = f_C = \frac{k}{p^{\text{DT}}},$$

and market clearing implies the equilibrium price is:

$$p^{\text{DT}} = (1+c)k + bL + R^{\text{ND}} = p^{\text{LF}} + R^{\text{ND}}.$$

Subsidies (SB). Under SB, the revenue R^{ND} is used to subsidize essential-good purchases at rate τ^{SB}. If p^{SB} is the price paid by consumers, sellers receive $(1+\tau^{\text{SB}})p^{\text{SB}}$. The planner's budget constraint is:

$$p^{\text{SB}}\tau^{\text{SB}}(bf_B + cf_C) = R^{\text{ND}}.$$

General equilibrium market clearing yields the consumer price:

$$p^{\text{SB}} = \left(\frac{1}{1+\tau^{\text{SB}}} + c\right)k + bL = p^{\text{LF}} - \left(1 - \frac{1}{1+\tau^{\text{SB}}}\right)k.$$

Remark 5. Parallel expressions can be derived for the case of distortionary taxation ($\tau > 1$).

Both regimes increase poor consumption relative to laissez–faire, but through different mechanisms. **Direct transfers** raise the equilibrium price for *all* agents. This reduces landlord and trader consumption, releasing more supply for the poor. **Subsidies**, in contrast, lower the consumer price for poor and traders while raising the price received by the landlord. This preserves high trader demand and can crowd out redistribution intended for the poor.

Lemma 1.1 (Direct transfers dominate subsidies for the poor)**.** *Fix $R, k, L > 0$ with $k > L$. Then for any $b > 0$ and any $c > 0$, the equilibrium consumption of the poor is strictly higher under direct transfers (DT) than under subsidies on the essential good (SB).*

Proof Idea. The key difference is how a fixed revenue R is channeled through prices. Under DT, the entire revenue is handed directly to the b poor. This raises the food price for *all* agents and compresses both landlord and trader demand; the resulting reduction in rich agents' consumption is fully absorbed by the poor. Under SB, by contrast, the same R is used to lower the consumer price of food for *both* poor and traders. Traders can still finance their food purchases from luxury wealth, so a non-trivial share of the extra subsidized supply is captured by them whenever $c > 0$. For any given (b, c, R, k, L), the equilibrium allocation thus allocates strictly more food to the poor under DT than under SB. (A formal proof is in Appendix A.3.1; simulations are presented in Figure 3.)

Lemma 1.2. *The difference in utilitarian welfare (total welfare accounting for traders and landlords) between SB and DT converges to zero as b becomes large.*

The proof can be found in the appendix A.3.2. This occurs because, with many poor agents, redistribution through either channel primarily reallocates food away from a fixed mass of rich traders and landlords.

We have thus established two results. First, for any fixed amount of revenue, direct transfers strictly dominate subsidies in improving poor consumption. Second, the aggregate welfare difference between these two redistribution methods vanishes in large economies. In contrast, we will show going forward that the welfare gap between spheres of exchange and transfer-based redistribution remains strictly positive as the mass of the poor grows. This implies that SOE also dominates subsidy-based redistribution by transitivity, and highlights that restricting convertibility is fundamentally distinct from redistributive taxation.

Comparison of Welfare Under SOE and Direct Transfers

We compare aggregate welfare under spheres of exchange (SOE) and commodity taxation with direct transfers (CT–DT). The welfare ranking depends critically on whether taxation operates in a non-distortionary or distortionary regime. Under non-distortionary taxation ($\tau \leq 1$), CT–DT raises revenue without deadweight loss but fails to restrict traders' demand for essential goods. Under distortionary taxation ($\tau > 1$), CT–DT restricts trader demand but does so by imposing allocative inefficiencies in the luxury sphere. SOE, by contrast, restricts traders' access to essential goods without generating deadweight loss, but raises no fiscal revenue.

Proposition 4. *Fix $k, L, c > 0$ with $k > L$. Consider a sequence of economies indexed by the number of landless laborers b, in which traders' luxury endowment is given by $S(b)$. Suppose that*

$$\lim_{b\to\infty} \frac{S(b)}{b} = 0 \qquad \textit{and} \qquad \lim_{b\to\infty} S(b) = \infty.$$

Then there exists $\widehat{B} = \widehat{B}(k, L, c)$ such that for all $b \geq \widehat{B}$, aggregate welfare under spheres of exchange satisfies

$$W_b^{\text{SOE}} > W_b^{\text{CT-DT}}.$$

Intuition. The welfare difference $W^{\text{SOE}} - W^{\text{DT}}$ is *non-monotonic* in the luxury endowment S, and this non-monotonicity drives the result.

When S is *very large* ($\lim_{b\to\infty} S(b)/b > 0$), low non-distortionary taxes ($\tau \leq 1$) generate substantial per-capita revenue without distorting luxury trade. This enables CT-DT to finance sizable transfers while preserving efficient luxury allocations. Indeed, if $S(b) \geq (k-L)b$, non-distortionary taxation achieves a per-capita transfer of $(k-L)$, replicating the first-best allocation of both essentials and luxuries.

When S is very small ($\lim_{b\to\infty} S(b) \leq \frac{4k}{L}$), high distortionary taxes ($\tau > 1$) effectively extract the limited luxury wealth while simultaneously restricting traders' essential-good demand. The deadweight loss from freezing the small luxury market is minor, allowing CT-DT to outperform SOE.

The pivotal case arises for intermediate values of $S(b)$. For large b, an intermediate S is neither large enough for non-distortionary taxation to generate meaningful per-capita revenue, nor small enough for distortionary taxation to be harmless. In this region, CT-DT faces a dilemma: it either (i) fails to restrict trader demand (under $\tau \leq 1$), or (ii) restricts demand only by imposing deadweight losses that cannot be offset by limited revenue (under $\tau > 1$). SOE, by contrast, restricts trader demand *without* any deadweight loss, directly increasing the poor's consumption of essentials. Consequently, for large b, SOE strictly dominates CT-DT over a nontrivial interval of luxury endowments S.

A formal proof is provided in Appendix A.3.3. Figure 4 illustrates the non-monotonic relationship between $S(b)$ and the welfare dominance of SOE without relying on asymptotics.

Generalization and Robustness: SOE vs. CT

Theorem 2. *Fix $L > 0$, $c > 0$, and $\sigma < 0$, and let $\gamma_1 > \gamma_2 > 0$. Consider a sequence of economies indexed by the number of landless laborers $b \in \mathbb{N}$. In the economy with b landless laborers, luxury endowments are given by a function $S : \mathbb{N} \to \mathbb{R}_+$, so that each trader is endowed with $(S(b), S(b))$ units of (s_1, s_2), and the landlord holds a normalized foodgrain endowment 1. Preferences in all these economies are*

$$U_1(f, l, s_1, s_2) = \frac{f^\sigma}{\sigma} + l + \gamma_1 s_1 + \gamma_2 s_2, \tag{2}$$

$$U_2(f, l, s_1, s_2) = \frac{f^\sigma}{\sigma} + l + \gamma_2 s_1 + \gamma_1 s_2, \tag{3}$$

with $\gamma_1 > \gamma_2 > 0$. Assume that S satisfies[15]

$$\lim_{b\to\infty} \frac{S(b)}{b} = 0 \qquad \textit{and} \qquad \lim_{b\to\infty} \frac{S(b)}{b^{\max\{\frac{1}{\sigma-1}-2\sigma,\, -\sigma\}}} = \infty.$$

[15]The second condition ($\lim_{b\to\infty} \frac{S(b)}{b^{\max\{\frac{1}{\sigma-1}-2\sigma,\, -\sigma\}}} = \infty$) ensures that, under distortionary taxation, deadweight losses in the luxury sphere asymptotically dominate any redistributive gains.

Then there exists $\widehat{B} = \widehat{B}(L, c, \sigma, \gamma_1, \gamma_2)$ such that for all $b \geq \widehat{B}$, aggregate welfare under spheres of exchange (W_b^{SOE}) exceeds aggregate welfare under any linear commodity-taxation regime with direct transfers (W_b^{CT}):

$$W_b^{\text{SOE}} > W_b^{\text{CT}}.$$

Proof Idea. Our argument mirrors lemma 1.1 and Proposition 4. First, we show that even with these *CES-type* preferences direct transfers are better than subsidies (Lemma 4.7). Then we show when S is not too large ($\lim_{b\to\infty} S(b)/b = 0$), non-distortionary CT cannot generate enough revenue to outperform SOE (Lemma 4.8). When S satisfies the lower bound, distortionary taxation creates deadweight losses that outweigh redistributive gains (Lemmas 4.9, 4.10, 4.11). Thus, for intermediate S values satisfying both conditions, CT cannot simultaneously generate revenue and avoid deadweight loss. A formal proof is in Appendix A.4; Figure 5 illustrates the result for various $\sigma < 0$.

Discussion of Assumptions. Our analysis relies on a small set of structural assumptions that serve to isolate the redistributive role of convertibility restrictions. First, we consider environments in which essential goods exhibit inelastic demand ($\sigma < 0$), a property consistent with historical and contemporary evidence for staple commodities. Second, we allow luxury wealth to grow with the size of the economy but at a rate slower than the mass of the poor. This rules out the case in which luxury wealth sufficiently abundant to finance redistribution through non-distortionary taxation. Third, we ensure that $S(b)$ doesn't grow too slowly. $\lim_{b\to\infty} \frac{S(b)}{b^{\max\{\frac{1}{\sigma-1}-2\sigma, -\sigma\}}} = \infty$ ensures that, under distortionary taxation, deadweight losses in the luxury sphere asymptotically dominate any redistributive gains. Together, these conditions highlight that intermediate values of surplus is where SOE design matters. Finally, our large-b asymptotics reflect the normative relevance of access to essential goods in societies with substantial inequality: as the mass of poor agents grows, welfare comparisons are driven by institutions that affect subsistence consumption.

We now turn to *Partial Spheres of Exchange* (PSOE), in which convertibility across spheres is neither fully prohibited nor fully unrestricted. The conceptual motivation for such intermediate regimes comes directly from historical settings in which exchange spheres were deliberately made permeable but costly to traverse. In ancient Mesopotamia, for example, a dual-currency system operated in which barley functioned as the medium for subsistence transactions and silver for long-distance and luxury trade, with conversion between the two regulated by the state at a fixed rate (Powell, 1996; Cripps, 2017). Similarly, among the Tiv of Nigeria, social norms sustained a restricted boundary between subsistence and prestige goods, permitting conversion only under limited and socially sanctioned circumstances (P.

Bohannan, 1959). These arrangements illustrate that partial restrictions on convertibility were both common and institutionally feasible.

Beyond their historical relevance, PSOEs are motivated by modern policy constraints. While our earlier results show that full SOEs can dominate commodity taxation under well-defined conditions, a wholesale replacement of the tax system is rarely feasible in contemporary economies. This motivates the study of PSOEs as hybrid institutions that complement rather than replace taxation: by combining restricted convertibility with fiscal instruments, PSOEs retain the demand-disciplining advantages of SOEs while preserving the revenue-generating capacity of taxes. As we show below, these instruments are complements rather than substitutes.

3.3 Partial Spheres of Exchange (PSOE)

The SOE regime fully prohibits conversion of luxury goods into essentials. While this benchmark clarifies the benefits of separating subsistence from luxury trade, policymakers may prefer intermediate arrangements. Allowing limited cross-sphere conversion—subject to a tax or exchange rate—can generate revenue while partially restricting the trader demand that bids up essential-good prices. We refer to such systems as *Partial Spheres of Exchange* (PSOE).

Let an agent choose an amount $I \geq 0$ of luxury-sphere income to convert into essential-sphere income. Let $\kappa \geq 0$ denote the exchange rate: converting I units of luxury purchasing power costs $(1+\kappa)I$ units of luxury goods. Let r denote per-capita transfers financed from PSOE revenue, and let $\mathbb{I}_{I>0}$ be the indicator equal to 1 if the agent converts across spheres. The PSOE budget constraints are

$$p_f(f - \hat{f}) + p_l(l - \hat{L}) \;\leq\; r + I,$$

$$p_{s_1}(s_1 - \hat{S}_1) + p_{s_2}(s_2 - \hat{S}_2) + (1 + \kappa \mathbb{I}_{I>0})I \;\leq\; 0.$$

When $\kappa = 0$, the constraints collapse to the laissez–faire budget set. When $\kappa \to \infty$, conversion is effectively prohibited and the SOE allocation is replicated. For intermediate κ, the economy lies between these two extremes. We continue to assume the preference structure introduced earlier (equation 2), which generates gains from trade in the luxury sphere and allows a tractable analytical characterization.

Proposition 5 (Optimal Exchange Rate under Partial Spheres of Exchange)**.** *Let social welfare be utilitarian. Consider a Partial Spheres of Exchange (PSOE) regime in which agents may convert luxury-sphere purchasing power into essential-sphere purchasing power* *at exchange rate $\kappa \geq 0$.Let λ_A, λ_B, and λ_C denote the marginal utilities of essential-sphere income for landlords, landless agents, and traders, evaluated at the PSOE equilibrium. Let p denote the equilibrium price of the essential good, and let $I > 0$ denote equilibrium cross-sphere conversion by traders.*[16]

At any interior optimum $\kappa > 0$,[17] *the optimal exchange rate satisfies*

$$\underbrace{\frac{dr}{d\kappa}(b\lambda_B)}_{\text{Revenue Effect}} - \underbrace{\frac{dp}{d\kappa}\Big[a\lambda_A(f_A - \hat{f}_A) + b\lambda_B f_B + c\lambda_C f_C\Big]}_{\text{Price Effect}} - \underbrace{c\,\frac{\lambda_C}{1+\kappa}\,I}_{\text{Trader Welfare Loss}} = 0.$$

Budget balance implies

$$r = \frac{c}{b}\kappa I, \qquad \frac{dr}{d\kappa} = \frac{c}{b}I + \frac{c}{b}\kappa\frac{dI}{d\kappa}.$$

Substituting into the first-order condition yields the equivalent expression

$$\left[\frac{c}{b}I + \frac{c}{b}\kappa\frac{dI}{d\kappa}\right](b\lambda_B) - \frac{dp}{d\kappa}\Big[a\lambda_A(f_A - \hat{f}_A) + b\lambda_B f_B + c\lambda_C f_C\Big] - c\,\frac{\lambda_C}{1+\kappa}\,I = 0.$$

Raising the exchange rate κ makes it more costly for traders to convert luxury purchasing power into essential goods, thereby reducing their effective demand for essentials. This directly reduces trader welfare, but it also raises revenue for redistribution and dampens trader demand in the essential-good market, lowering foodgrain prices. The optimal exchange rate balances these competing effects. A full derivation is in Appendix A.5.

Joint Policies and Complementarity

We now consider an environment in which the planner can use *both* partial spheres of exchange (PSOE) and commodity taxation (CT). Let τ denote the ad valorem tax rate on luxury transactions and κ the exchange rate charged when agents convert resources from the luxury sphere into the essential sphere. As before, luxury endowments are symmetric across goods, $\hat{S}_1^C = \hat{S}_2^C = S$, and preference types are symmetrically distributed. Hence the planner always chooses a common tax rate on both luxury goods, denoted τ. Also, under symmetry, we write $p_{s_1} = p_{s_2} \equiv p_s$.

Let R denote the *total revenue* collected by the planner from both instruments (commodity taxes and exchange–rate charges). As before, this revenue is transferred uniformly

[16] Since only traders hold luxury endowments in equilibrium, conversion decisions are relevant only for traders.

[17] Corner solutions at $\kappa = 0$ and $\kappa \to \infty$ correspond to laissez-faire and SOE, respectively, and are already characterized above.

to the b poor agents, so each poor agent receives R/b. The essential–sphere budget constraint therefore includes the term R/b for poor agents only.

An agent's budget constraints under joint policies are:

$$p_f\big(f - \hat{f}\big) + p_l\big(l - \hat{L}\big) \leq I + T$$

$$\frac{p_s}{1 + \tau \mathbb{1}_{\{S - s_1 \geq 0\}}}(s_1 - \hat{S}) + \frac{p_s}{1 + \tau \mathbb{1}_{\{S - s_2 \geq 0\}}}(s_2 - \hat{S}) + (1 + \kappa \mathbb{1}_{\{I \geq 0\}})I \leq 0$$

where I denotes the amount of luxury–sphere income an agent chooses to convert into essential–sphere income. Luxury sales are taxed at rate τ, generating a wedge between the buyer and seller price. We write $T := R/b$ for the per-capita transfer received by each poor agent and for non-poor agents $T := 0$. Total revenue consists of commodity-tax receipts on luxury sales and exchange-rate charges κI collected from traders. The exchange–rate charge κ applies only when $I > 0$.

Theorem 3 (CT complements PSOE)**.** *Suppose preferences are given by equations (2). Consider an economy with a positive exchange rate $\kappa > 0$ across spheres and no commodity taxes ($\tau = 0$). Let the equilibrium in the economy be such that the poor are still income constrained in the essential sphere such that any marginal increase in income is only spent on essentials. A small increase in the commodity tax rate τ, with all associated revenue rebated to the poor, strictly increases the welfare of the poor, and strictly increases utilitarian welfare.*

Proof Idea. For sufficiently small τ, the luxury sphere remains undistorted: all luxury endowments continue to be supplied and traded efficiently, so a marginal increase in τ raises revenue without affecting the allocation of luxuries. The resulting revenue is rebated lump-sum to the poor and is spent entirely on essentials. The increased demand from the poor cannot be absorbed at the initial price: instead, the equilibrium food price p must rise. Since food demand of landlords and rich traders is downward sloping in p, their food consumption falls. By market clearing, this releases food to the poor, raising their consumption and increasing utilitarian welfare because the marginal utility for essentials is higher for the poor. A formal derivation is provided in Appendix A.6.1.

Theorem 4 (PSOE complements commodity taxation)**.** *Suppose preferences are given by equations (2). Fix any commodity tax rate $\tau > 0$ and consider an allocation with free conversion across spheres ($\kappa = 0$). Let the equilibrium in the economy be such that the poor are still income constrained in the essential sphere such that any marginal increase in income* *is only spent on essentials. Then there exists a pair* (τ', κ') *with* $\tau' > 0$ *and* $\kappa' > 0$ *such that utilitarian welfare is weakly higher under* (τ', κ') *than under* $(\tau, 0)$.

Proof Idea. The proof proceeds by cases. When τ is non-distortionary, the result follows directly from Theorem 3: introducing a positive exchange rate $\kappa > 0$ raises revenue without distorting luxury trade and strictly increases poor welfare.

When τ is distortionary, we construct an alternative joint policy (τ', κ') with $\tau' > 0$ and $\kappa' > 0$ that weakly increases poor welfare relative to $(\tau, 0)$. In particular, the planner reduces τ to its maximum non-distortionary level and simultaneously increases κ so as to weakly increase the effective price faced by traders and landlords in the essential-good market. This policy ensures that the revenue weakly increases while eliminating the deadweight loss associated with excessive commodity taxation, preserves efficiency in the luxury spheres and ensures that the poor people's essential consumption doesn't fall. Together ensuring that the utilitarian welfare is weakly higher. Appendix A.6.2 provides the formal argument.

These results establish that the two instruments are complements rather than substitutes: commodity taxes efficiently generate revenue at low rates, while exchange–rate charges restrict luxury-to-essential conversion without distorting within-sphere trade. An optimal policy therefore uses both instruments jointly.

4 Implementing Spheres of Exchange

This section discusses how spheres of exchange (SOEs) can be implemented and sustained. We first consider the informational and enforcement requirements associated with maintaining distinct trading spheres. We then draw on historical evidence from tribal and ancient economies to illustrate how such systems operated in practice. Finally, we discuss how the logic of SOEs helps interpret contemporary redistributive institutions and policy debates.

4.1 Information and Enforcement Cost

Implementing spheres of exchange requires information about aggregate endowments and preferences in order to assess whether restricting convertibility between goods improves welfare relative to laissez–faire. This informational requirement is the same to that faced by a planner designing optimal commodity taxes or rationing schemes.

The central operational challenge is enforcement: maintaining separation between essential and luxury spheres. In a full SOE, direct exchange across spheres is prohibited; in a partial SOE, conversion is permitted only at a regulated rate that overvalues essential-sphere purchasing power. Such restrictions create incentives for evasion, including informal barter or black-market exchange. These challenges are not unique to SOEs. Commodity taxes face evasion (Allingham and Sandmo, 1972; Cremer and Gahvari, 1993; Slemrod, 2007; Guyton et al., 2021), while price controls and rationing invite illicit trade (Galbraith, 1946; Cox, 1980; Gadenne, 2020). From a theoretical perspective, there is no basis to presume that enforcing SOEs entails greater informational or enforcement costs than conventional redistributive instruments.[18]

4.2 Enforcement in Tribal and Ancient Economies

In small-scale societies, separation between exchange spheres was typically sustained through social norms rather than formal monitoring. Among the Tiv of Nigeria, prestige goods such as brass rods were not ordinarily used to purchase food; violations were socially sanctioned except in extreme circumstances (P. Bohannan, 1959). Similar norms governed other stateless societies. Among the Fur of Sudan, agricultural labor could be compensated only with beer rather than money, and selling beer for cash was considered morally unacceptable (Barth, 1967). In the Trobriand Islands, ceremonial *kula* valuables circulated in a closed network governed by reciprocity and taboo and could not be exchanged for everyday goods (Malinowski, 1922). Across these contexts, moral and ritual sanctions limited the conversion of wealth into subsistence consumption, preserving access to essential goods without centralized enforcement.

In more complex economies, similar objectives were implemented through formal institutions. In ancient Mesopotamia, barley functioned as the primary medium for wages and subsistence transactions, while silver was used for long-distance and luxury trade (Powell, 1996). Conversion between the two was regulated by the state (Cripps, 2017), limiting arbitrage and stabilizing access to essentials. Whether enforced through social norms or bureaucratic control, these historical systems demonstrate that restricting convertibility across goods was institutionally feasible across a wide range of economic environments.

4.3 Are spheres of exchange relevant in modern economies?

We now discuss how the logic of spheres of exchange can be used to interpret contemporary redistributive institutions in two modern economic settings.

[18]In reality, these costs might significantly differ across regimes and should be explicitly taken into account while designing policy.

4.3.1 SOEs and the Public Distribution System in India

India's Public Distribution System (PDS) is one of the largest food-rationing programs in the world, providing subsidized foodgrains to low-income households under the National Food Security Act and related schemes (Press Information Bureau, Government of India, 2024). While entitlements are defined in terms of quantities, a distinctive feature of the PDS is that the type of grain distributed—rice, wheat, or occasionally coarse grains—is fixed at the state level, reflecting procurement patterns and historical consumption norms (Press Information Bureau, Government of India, 2020). This design interacts in important ways with internal migration. Although the One Nation One Ration Card (ONORC) reform allows beneficiaries to access their rations anywhere in the country, portability applies only to quantities, not to grain type. As a result, migrants must consume the grain supplied in the destination state—even when it differs from their preferred staple—potentially reducing the effective welfare gains from the program, especially among households with strong dietary preferences (S. I. Rajan, S. Rajan, and M., 2020; Gupta et al., 2023).

Viewed through the lens of our framework, this feature of the PDS represents a within-sphere choice constraint. The program successfully restricts access to essential goods—thereby achieving its redistributive objective—but limits individual choice within the subsistence sphere.[19] Our results show that when preferences over essential goods are heterogeneous, relaxing such within-sphere constraints can raise welfare without undermining equity in access. Interpreted through this lens, policies that allow greater flexibility in grain choice at the point of redemption would preserve the core redistributive structure of the PDS while improving allocative efficiency. Moreover, by reducing consumption mismatches for migrants, such flexibility may also relax migration frictions, amplifying welfare gains beyond the subsistence sphere (Gelb and Mukherjee, 2020).

4.3.2 SOEs and Automation

Our analysis highlights that spheres of exchange (SOEs) outperform taxation and transfers when a large fraction of the population relies primarily on selling leisure, and when surplus from luxury trade is positive but at intermediate levels. We can also interpret luxury endowments S as specialized skills or human capital, a condition relevant in contemporary economies, where human capital is a dominant source of income and inequality. Rapid

[19] In the terminology of our framework, the PDS corresponds to a form of *partial rationing*: households receive a fixed in-kind entitlement of a specific foodgrain (at zero or subsidized prices), but remain free to purchase additional quantities from the market at prevailing prices. Our SOE framework highlights that, while such entitlements restrict access to essentials at the subsistence margin, welfare can be further improved by allowing flexibility in the *type* of foodgrain consumed within the essential-goods sphere.

advances in artificial intelligence (AI) and automation are reshaping these endowment structures as documented by the growing literature that documents task automation reduces labor demand and the labor share (Acemoglu and Restrepo, 2019), with occupations exposed to AI-facing tasks experiencing measurable employment declines (Hampole et al., 2025). These developments can create environments in which a large mass of agents relies primarily on selling undifferentiated labor, while a smaller group earns rents from specialized skills—precisely the setting in which SOEs are welfare-improving in our framework.

Automation operates through two distinct channels. The *displacement effect* arises when machines substitute for skilled labor, reducing effective luxury endowments S while increasing the mass of agents endowed only with leisure. The *erosion effect* occurs when automation lowers the market value of routine or unskilled labor, reducing the subsistence endowment L without directly affecting skilled income. Our model predicts sharply different policy rankings across these cases. When the displacement effect is such that skilled rents decline, vanishing for some and not vanishing for some—SOEs improve welfare by restricting high-income demand for essentials while preserving efficiency in non-essential markets (as documented in Theorem 2). By contrast, when the erosion effect dominates ($L \to 0$), taxation with direct transfers strictly dominates SOEs, as convertibility restrictions can no longer sustain positive subsistence consumption for the unskilled. This result is formalized as Proposition 6 in the Appendix.

4.3.3 Implementation of SOEs in Modern Economies

Spheres of exchange (SOEs) can be operationalized in contemporary economies either through a dual monetary system or through expenditure-based constraints on essential goods. A modern analogue of historical dual-currency regimes would involve two fiat-based media of exchange—one designated for essential transactions and another for nonessential or luxury spending—linked by a government-determined exchange rate that permits limited, regulated conversion across spheres. A more practical and institutionally familiar alternative is the use of *expenditure ceilings* on essential goods, which cap the maximum amount an individual can spend within the essential-goods sphere. When set at the Rawlsian income—the consumption level of the least advantaged—such ceilings replicate the SOE allocation characterized in Section 3 by curbing high-income demand for essentials while preserving market allocation within spheres.[20] Advances in digital payments, identification systems, and transaction monitoring potentially make these instruments more feasible than in the past, allowing real-time enforcement of expenditure limits with limited administrative burden, subject to

[20]The term "Rawlsian income" follows the social-choice literature; see https://plato.stanford.edu/entries/social-choice/#DocParDisDil.

institutional capacity and appropriate data-protection safeguards.[21] Properly designed, such technology-enabled SOEs preserve redistributive intent while leveraging modern enforcement capacity.

5 Conclusion

This paper develops a theoretical foundation for spheres of exchange (SOEs)—a well-documented institution in ancient and tribal economies—and shows how they can function as effective redistributive mechanisms. By segmenting trade into subsistence and luxury spheres, SOEs preserve allocative efficiency within each sphere while improving access to essential goods for poorer agents. In contrast to standard fiscal instruments, redistribution is achieved through restrictions on convertibility rather than through taxes, transfers, or quantity controls.

Within a general-equilibrium framework, we show that SOEs dominate quantity rationing when preferences over essential goods are heterogeneous and aggregate endowments are symmetric, because they allow prices to allocate goods efficiently within the subsistence sphere. When luxury goods are heterogeneous and generate gains from trade, SOEs avoid the deadweight losses inherent in commodity taxation by restricting only cross-sphere conversion while preserving efficiency within the luxury sphere. When the surplus from luxury trade lies in an intermediate range, SOEs strictly dominate commodity taxation: they dampen high-income demand for essentials without distorting luxury–luxury exchange. We further show that *partial* spheres of exchange—implemented through a regulated conversion between luxury and subsistence spheres—can complement low commodity taxes, allowing limited cross-sphere flows while preserving equitable access to essentials.

Beyond explaining the persistence of SOEs across diverse historical settings, our framework provides a template for modern redistributive design. The combination of within-sphere efficiency and cross-sphere restriction speaks directly to contemporary policy questions, including the design of rationing systems with heterogeneous preferences, portability of in-kind transfers, and the distributional consequences of automation. More broadly, the analysis contributes to the emerging literature on inequality-aware market design by identifying institutional constraints on trade—rather than prices or quantities—that improve welfare when conventional fiscal instruments offer limited scope for improvement.

Several questions remain open. While we provide a rationale for two-sphere systems in economies with two broad classes of goods, the optimal number, structure, and interaction of spheres in richer environments is an important direction for future work. Our analysis also

[21]For example, expenditures on essential goods could be linked to unique identifiers such as Social Security numbers in the U.S. or Aadhaar in India.

abstracts from production, political economy, and enforcement frictions in order to isolate the core redistributive mechanism of restricted convertibility. Extending the framework to settings with endogenous income generation, heterogeneous enforcement technologies, or multiple interacting spheres may yield further insights into when SOE-like institutions emerge and how they compare with standard fiscal tools. We view spheres of exchange as a promising object of study at the intersection of market design, public finance, and economic history.

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A Appendix

A.1 3 Good Economy: Welfare Comparison

We now compare social welfare under the laissez-faire (LF) and spheres of exchange (SOE) regimes. Since the luxury-good market remains undistorted in both regimes, welfare differences arise solely from the foodgrain–leisure sphere. Aggregate utilitarian welfare under SOE can be written as:

$$W^{SOE} = (b+c)k \ln\left[\frac{L}{(b+c)L+k}\right] + k \ln\left[\frac{k}{(b+c)L+k}\right]. \tag{4}$$

Similarly, welfare under laissez-faire is:

$$W^{LF} = bk \ln\left[\frac{L}{bL+k(1+c)}\right] + (1+c)k \ln\left[\frac{k}{bL+k(1+c)}\right]. \tag{5}$$

The SOE regime yields higher welfare than laissez-faire if and only if

$$\begin{aligned}(b+c)k \ln\left[\frac{L}{(b+c)L+k}\right] &+ k \ln\left[\frac{k}{(b+c)L+k}\right] \\ \geq\ bk \ln\left[\frac{L}{bL+k(1+c)}\right] &+ (1+c)k \ln\left[\frac{k}{bL+k(1+c)}\right],\end{aligned}$$

which simplifies to

$$c\ \ln\left(\frac{L}{k}\right) + (b+c+1)\ \ln\left(\frac{bL+k(1+c)}{(b+c)L+k}\right) \geq 0$$
$$\iff \quad \left(\frac{bL+k(1+c)}{(b+c)L+k}\right)^{\frac{b+c+1}{c}} \geq \frac{k}{L}.$$

Taking limits as $b \to \infty$ shows that

$$\lim_{b\to\infty} \left(\frac{bL+k(1+c)}{(b+c)L+k}\right)^{\frac{b+c+1}{c}} = e^{\frac{k}{L}-1} > \frac{k}{L},$$

for any $k > L$, confirming that there is some number $\hat{B} = B(L, k, S)$ such that for all $b > \hat{B}$, utilitarian welfare $W = U^A + bU^B + cU^C$ is more under SOE than LF.

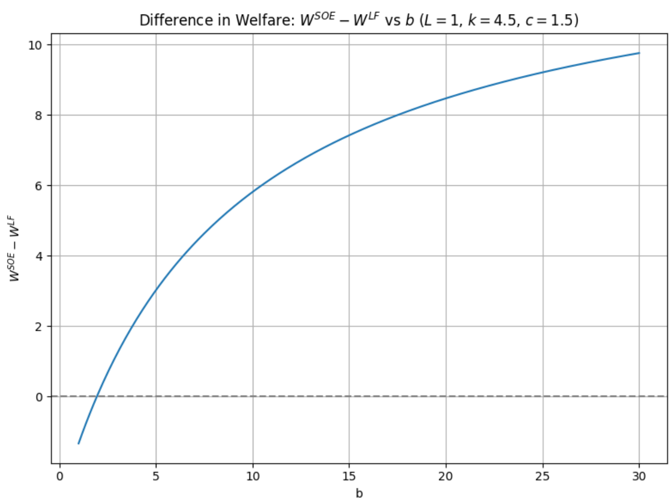

Figure 1: Welfare Difference (ΔW) between SOE and laissez-faire for the poor on the y-axis and relative number of poor on the x-axis (b). Utility functions for all agents: $klnf+l+s$ where $k = 4.5$. Endowment of leisure for everyone is $L = 1$. Relative number of rich are $c = 1.5$. Relative number of landowners $a = 1$.

A.2 The Case Against Rationing: Preference Heterogeneity

Type	**Barley** (f_1)	**Wheat** (f_2)	**Leisure** (l)	**Silver** (s)
Landowner	1/2	1/2	L	0
Landless	0	0	L	0
Trader	0	0	L	S

Table 3: Endowment distribution with two essential goods.

Endowment Type \ Preference Type	U_1	U_2
Landowner	$\left(\alpha\frac{k}{p^{SOE}}, (1-\alpha)\frac{k}{p^{SOE}}\right)$	$\left((1-\alpha)\frac{k}{p^{SOE}}, \alpha\frac{k}{p^{SOE}}\right)$
Landless	$\left(\alpha\frac{L}{p^{SOE}}, (1-\alpha)\frac{L}{p^{SOE}}\right)$	$\left((1-\alpha)\frac{L}{p^{SOE}}, \alpha\frac{L}{p^{SOE}}\right)$
Trader	$\left(\alpha\frac{L}{p^{SOE}}, (1-\alpha)\frac{L}{p^{SOE}}\right)$	$\left((1-\alpha)\frac{L}{p^{SOE}}, \alpha\frac{L}{p^{SOE}}\right)$

Table 4: Consumption of (f_1, f_2) under SOE. The price $p^{SOE} = (b+c)L + k$ is determined in general equilibrium.

Agent— Pref	U_1	U_2
Landowner	$\left(\alpha\frac{k}{p}, (1-\alpha)\frac{k}{p}\right)$	$\left((1-\alpha)\frac{k}{p}, \alpha\frac{k}{p}\right)$
Landless	$\left(\min\left\{\alpha\frac{L}{p}, \bar{q}\right\}, \min\left\{(1-\alpha)\frac{L}{p}, \bar{q}\right\}\right)$	$\left(\min\left\{(1-\alpha)\frac{L}{p}, \bar{q}\right\}, \min\left\{\alpha\frac{L}{p}, \bar{q}\right\}\right)$
Trader	$\left(\min\left\{\alpha\frac{k}{p}, \bar{q}\right\}, \min\left\{(1-\alpha)\frac{k}{p}, \bar{q}\right\}\right)$	$\left(\min\left\{(1-\alpha)\frac{k}{p}, \bar{q}\right\}, \min\left\{\alpha\frac{k}{p}, \bar{q}\right\}\right)$

Table 5: Demand for (f_1, f_2) under symmetric QR

Proof of Proposition 3

Proof. Step 1: SOE achieves the first-best allocation among the poor. Under SOE, equal aggregate endowments and equal population shares of the two preference types imply $p_{f_1} = p_{f_2} = p$. A poor agent (with essential–sphere income $m = L$) of type α therefore attains indirect utility

$$V_\alpha(p, m) = \alpha k \ln \frac{\alpha m}{p} + (1-\alpha) k \ln \frac{(1-\alpha) m}{p},$$

which satisfies

$$\frac{\partial V_\alpha}{\partial m} = \frac{k}{m},$$

independent of α. Thus all preference types have the same marginal utility of income.

Since all poor agents have the same marginal utility of income, their utilitarian welfare—holding aggregate income fixed—is maximized by equalizing incomes (a standard implication of concavity; see also (Negishi, 1960)). In our setting, the SOE regime does exactly this for landless laborers: they have identical leisure endowments and no access to silver, so they enter the essential sphere with the same income L. Hence, among all feasible allocations of essential–sphere income across poor agents (holding their total income fixed), the SOE allocation is welfare-maximizing for the poor.

Step 2: QR cannot improve on the average SOE allocation for the poor. Under any QR policy, rich agents have weakly higher essential–sphere income than poor agents. Under log preferences, each foodgrain is strictly normal, their average consumption must satisfy

$$\bar{f}_C^{\mathrm{QR}} \;\geq\; \bar{f}_B^{\mathrm{QR}} \qquad \text{for each essential good.}$$

A poor agent of type α with income L facing a common price p^{SOE} over barley and wheat chooses

$$f_1 = \frac{\alpha L}{p^{SOE}}, \qquad f_2 = \frac{(1-\alpha)L}{p^{SOE}},$$

Given α and $1-\alpha$ types are equally probable, the average consumption of the poor is

$$\bar{f}_{1,B}^{SOE} = \bar{f}_{2,B}^{SOE} = \frac{L}{2p^{SOE}},$$

under SOE.

Now consider any symmetric QR policy (as in Remark 3), so that the equilibrium foodgrain prices again satisfy

$$p_{f_1}^{\mathrm{QR}} = p_{f_2}^{\mathrm{QR}} = p^{\mathrm{QR}}.$$

Hence, under QR the poor face the price p^{QR} and have the same essential-sphere income L, so their total consumption of the two essentials satisfies

$$f_1 + f_2 \;\leq\; \frac{L}{p^{\mathrm{QR}}}.$$

Therefore, their average per-good consumption is bounded above by

$$\bar{f}_B^{\mathrm{QR}} \;\leq\; \frac{1}{2} \cdot \frac{L}{p^{\mathrm{QR}}}$$

Suppose, toward a contradiction, that the poor consume strictly more of (say) barley on average under QR than under SOE:

$$\bar{f}_B^{\text{QR}} > \bar{f}_B^{\text{SOE}}.$$

Because rich consumption cannot fall and aggregate endowments are fixed, feasibility then requires that landlords consume strictly less barley under QR than under SOE. Since barley is a normal good for the landlord as well, this can only occur if the common foodgrain price under QR exceeds the SOE price:

$$p^{\text{QR}} > p^{\text{SOE}}.$$

But with $p^{\text{QR}} > p^{\text{SOE}}$ and income L, we must have that the average consumption,

$$\bar{f}_{1,B}^{\text{QR}} \leq \frac{L}{2p^{\text{QR}}} < \frac{L}{2p^{\text{SOE}}} = f_{1,B}^{\text{SOE}}$$

and likewise for f_2. Hence the average barley (and wheat) consumption of the poor under QR is less than under SOE, contradicting the assumption $\bar{f}_B^{\text{QR}} > \bar{f}_B^{\text{SOE}}$.

Conclusion. No QR policy can deliver strictly higher poor welfare than SOE. Since SOE already maximizes the poor's welfare given their aggregate essential–sphere income, this implies that SOE weakly dominates QR for landless laborers. □

Proof of Theorem 1: General Dominance of SOE for Symmetric CES

We prove this as a series of claims.

Claim 4.1 (SOE dominates QR for the poor under symmetric CES preferences)**.** *Consider preference types $t \in \{1, 2\}$ with CES (constant elasticity of substitution) preferences over essential goods:*

$$U_1(f_1, f_2, l, s) = \alpha\frac{f_1^\sigma}{\sigma} + (1-\alpha)\frac{f_2^\sigma}{\sigma} + l + s,$$
$$U_2(f_1, f_2, l, s) = (1-\alpha)\frac{f_1^\sigma}{\sigma} + \alpha\frac{f_2^\sigma}{\sigma} + l + s,$$

where $\alpha \in [0,1]$ and $\sigma < 1$, $\sigma \neq 0$. Let the aggregate endowments of the two essentials coincide and assume that the two preference types occur with equal probability. Then SOE yields weakly higher welfare for the poor than any feasible QR policy.

Proof. We establish the result through three lemmas.

Lemma 4.1 (SOE achieves the first-best among the poor)**.** *Under SOE, for any given aggregate food budget for the poor, the resulting allocation is utilitarian first-best among them.*

Proof. Because aggregate endowments of barley and wheat are equal and the two preference types are equally likely, the SOE equilibrium in the essential sphere satisfies $p_1 = p_2 = p^{\text{SOE}}$.

A poor agent with essential-sphere income $m = L$ solves

$$\max_{f_1, f_2} \; \alpha \frac{f_1^\sigma}{\sigma} + (1-\alpha)\frac{f_2^\sigma}{\sigma} \quad \text{subject to} \quad pf_1 + pf_2 \leq L,$$

where $p = p^{\text{SOE}}$. The first-order conditions yield

$$\alpha f_1^{\sigma-1} = \lambda p, \qquad (1-\alpha) f_2^{\sigma-1} = \lambda p,$$

so

$$\frac{f_1}{f_2} = \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{1-\sigma}} =: r(\alpha).$$

Using the budget constraint $pf_1 + pf_2 = L$, the optimal demands are

$$f_1 = \frac{L\, r(\alpha)}{p[1 + r(\alpha)]}, \qquad f_2 = \frac{L}{p[1 + r(\alpha)]}.$$

The resulting indirect utility can be written as

$$V_\alpha(p, L) = \frac{L^\sigma}{\sigma p^\sigma [1 + r(\alpha)]^\sigma} \left[\alpha r(\alpha)^\sigma + (1-\alpha)\right].$$

The marginal utility of income is

$$\frac{\partial V_\alpha}{\partial L} = \frac{L^{\sigma-1}}{p^\sigma [1 + r(\alpha)]^\sigma} \left[\alpha r(\alpha)^\sigma + (1-\alpha)\right].$$

Define

$$R(\alpha) := \frac{\alpha r(\alpha)^\sigma + (1-\alpha)}{[1 + r(\alpha)]^\sigma}.$$

A direct computation, using the fact that $r(1-\alpha) = 1/r(\alpha)$, shows that $R(\alpha) = R(1-\alpha)$ for all α. Since our economy has exactly two preference types, indexed by α and $1-\alpha$, this implies that the marginal utility of income is identical for all poor agents in the SOE equilibrium.

Since all poor agents have the same marginal utility of income and indirect utility is concave in income, utilitarian welfare over the poor—holding their aggregate essential-sphere income fixed—is maximized by equalizing incomes across them (for the formal argument, see Negishi (1960)). In our setting, landless laborers have identical leisure endowments and no access to silver, so they enter the essential sphere with the same income L under SOE. Hence, among all feasible allocations of essential-sphere income across the poor, the SOE allocation is utilitarian first-best. □

Lemma 4.2 (Rich consume weakly more under QR)**.** *Under any QR regime, the average consumption of each essential good by the rich is weakly greater than that of the poor.*

Proof. Rich agents (traders and landowners) have strictly higher essential-sphere income than poor agents. The utility function is additively separable and strictly concave in f_1 and f_2 for $\sigma < 1$, $\sigma \neq 0$, so both goods are normal. Therefore, holding prices fixed, higher income yields weakly higher demand for each essential good. Averaging across individuals implies that, for each essential good, the rich consume at least as much on average as the poor. □

Lemma 4.3 (QR cannot increase the poor's food budget)**.** *No QR policy can deliver higher average essential-good consumption for the poor than the SOE allocation.*

Proof. Let $\bar{f}_B^{\text{SOE}}$ denote the average *per-good* consumption of a poor agent under SOE. With a single price p^{SOE} for both essentials and budget L, each poor agent has total consumption

$$f_1 + f_2 = \frac{L}{p^{\text{SOE}}}.$$

By symmetry of types and equal aggregate endowments, this implies that the average consumption per essential good is

$$\bar{f}_B^{\text{SOE}} = \frac{1}{2} \cdot \frac{L}{p^{\text{SOE}}}.$$

Suppose, for contradiction, that some QR policy yields $\bar{f}_B^{\text{QR}} > \bar{f}_B^{\text{SOE}}$ for at least one of the two goods. By Lemma 4.2, rich agents also consume at least as much of each essential as the poor, so

$$\bar{f}_C^{\text{QR}} \;\geq\; \bar{f}_B^{\text{QR}}.$$

Since aggregate endowments of each essential good are fixed, higher average consumption by both poor and rich must be offset by lower consumption by landowners. Because essentials are normal goods for landowners as well, their demand can fall only if the common foodgrain price rises. Hence the QR equilibrium must satisfy $p^{\text{QR}} > p^{\text{SOE}}$.

However, under QR the poor face the price p^{QR} and have the same essential-sphere income

L, so their total consumption of the two essentials satisfies

$$f_1 + f_2 \;\le\; \frac{L}{p^{\mathrm{QR}}}.$$

Therefore, their average per-good consumption is bounded above by

$$\bar{f}_B^{\mathrm{QR}} \;\le\; \frac{1}{2}\cdot\frac{L}{p^{\mathrm{QR}}} < \frac{1}{2}\cdot\frac{L}{p^{\mathrm{SOE}}} = \bar{f}_B^{\mathrm{SOE}},$$

contradicting the assumption that $\bar{f}_B^{\mathrm{QR}} > \bar{f}_B^{\mathrm{SOE}}$.

Hence, no QR policy can provide higher average essential-good consumption to the poor than the SOE allocation. □

Lemmas 4.1, 4.2, and 4.3 together establish that SOE yields weakly higher welfare for the poor than any feasible QR policy under symmetric CES preferences, which proves Claim 4.1. □

Discussion. This proof demonstrates that the advantage of SOEs over rationing is not specific to Cobb-Douglas preferences but extends to the entire class of symmetric CES preferences. The key insight is that when preferences are symmetric across goods and agents, SOEs simultaneously achieve both **distributive efficiency** (equalizing the marginal utility of income) and **allocative efficiency** (allowing optimal substitution), while QR faces an inescapable trade-off between these objectives.

The result highlights SOEs as a robust institutional solution for equitable distribution in economies with preference heterogeneity over essential goods.

Claim 4.2 (Large-b welfare dominance of SOE over QR)**.** *Consider preference types $t \in \{1,2\}$ with CES preferences over two essential goods:*

$$U_1(f_1, f_2, l, s) = \alpha\frac{f_1^\sigma}{\sigma} + (1-\alpha)\frac{f_2^\sigma}{\sigma} + l + s,$$
$$U_2(f_1, f_2, l, s) = (1-\alpha)\frac{f_1^\sigma}{\sigma} + \alpha\frac{f_2^\sigma}{\sigma} + l + s,$$

where $\alpha \in (0,1)$ and $\sigma < 0$. There are b landless labourers (the poor), c rich traders, and one landlord. All agents have leisure endowment $L > 0$.

Let utilitarian welfare in regime $R \in \{\mathrm{SOE}, \mathrm{QR}, \mathrm{LF}\}$ be

$$W_b^R \;=\; U^{A,R} + b\,U^{B,R} + c\,U^{C,R},$$

where $U^{A,R}, U^{B,R}, U^{C,R}$ denote landlord, representative poor, and representative rich utilities in regime R.

Under QR, the planner imposes a symmetric per-good cap $q \geq 0$ on net purchases:

$$f_i - \hat{f}_i \;\leq\; q, \qquad i \in \{1, 2\},$$

for all non-landlord agents. Let $W_b^{\mathrm{QR}}(q)$ denote the corresponding equilibrium welfare.

Then there exists $\widehat{B} = \widehat{B}(L, c, \sigma, \alpha)$ such that, for all $b \geq \widehat{B}$ and all $q \geq 0$,

$$W_b^{\mathrm{SOE}} \;>\; W_b^{\mathrm{QR}}(q).$$

The proof has two parts. First, among symmetric QR policies it suffices to consider caps that are slack for all poor agents and binding for some rich agents (Region II). Second, Lemma 4.4 and Lemma 4.5 compares SOE and Region II QR as $b \to \infty$.

Firstly, notice that the welfare effect of policy on the landlords depends only on price (as quantity caps never affect them), and their contribution to total welfare is monotonically decreasing in price. The only time which they prefer QR to SOE in terms of essential good consumption is when is when the cap

Region I: all caps slack. Suppose that the QR caps are slack for every agent in equilibrium. Then the QR allocation coincides with the laissez–faire (LF) allocation:

$$W_b^{\mathrm{QR}}(q) = W_b^{\mathrm{LF}}.$$

We claim that $W_b^{\mathrm{SOE}} > W_b^{\mathrm{LF}}$ for all sufficiently large b. Let $\sigma < 0$ and define

$$\phi := \frac{\sigma}{\sigma - 1} \in (0, 1).$$

Under LF, rich traders may spend their silver endowment $S > 0$ on essentials, while under SOE traders have essential–sphere income L. Let p^{LF} and p^{SOE} denote the essential-good prices. The average demand for all types under LF is $f_A = \frac{1}{2}\left[\frac{1}{\alpha}^{\frac{1}{\sigma-1}} + \frac{1}{(1-\alpha)}^{\frac{1}{\sigma-1}}\right](p^{LF})^{\phi-1}$; $f_B = \frac{L}{2p^{LF}}$; $f_C = min\{\frac{L+S}{2p}, f_A\}$.

The average demand for all types under SOE is $f_A = \frac{1}{2}\left[\frac{1}{\alpha}^{\frac{1}{\sigma-1}} + \frac{1}{(1-\alpha)}^{\frac{1}{\sigma-1}}\right](p^{SOE})^{\phi-1}$; $f_B = \frac{L}{2p^{SOE}}$; $f_C = \frac{L}{2p^{SOE}}$;

(i) Consider the case $\frac{L+S}{2p} < f_A$

Market clearing yields:

$$p^{\mathrm{LF}} - \left[\frac{1}{\alpha}^{\frac{1}{\sigma-1}} + \frac{1}{(1-\alpha)}^{\frac{1}{\sigma-1}}\right]\left(p^{\mathrm{LF}}\right)^{\phi} = (b+c)L + cS, \tag{6}$$

$$p^{\mathrm{SOE}} - \left[\frac{1}{\alpha}^{\frac{1}{\sigma-1}} + \frac{1}{(1-\alpha)}^{\frac{1}{\sigma-1}}\right]\left(p^{\mathrm{SOE}}\right)^{\phi} = (b+c)L. \tag{7}$$

Since $p - \left[\frac{1}{\alpha}^{\frac{1}{\sigma-1}} + \frac{1}{(1-\alpha)}^{\frac{1}{\sigma-1}}\right] p^{\phi}$ is increasing in p for large p, we have $p^{\mathrm{LF}} > p^{\mathrm{SOE}}$.[22]

Asymptotics. For large b, Taylor expansion yield:

$$p^{\mathrm{LF}} - p^{\mathrm{SOE}} = cS + O(b^{\phi-1}).$$

Welfare comparison. For a poor agent with essential-sphere income L,

$$U^{B,LF} = V(p_{LF}, L).$$

A first-order expansion around p^{SOE} yields

$$U^{B,\mathrm{SOE}} - U^{B,\mathrm{LF}} = V(p_{SOE}, L) - V(p_{LF}, L) = \frac{L^{\sigma}}{\sigma}\left[\alpha^{\frac{1}{1-\sigma}} + (1-\alpha)^{\frac{1}{1-\sigma}}\right]^{1-\sigma}\left[p_{SOE}^{-\sigma} - p_{LF}^{-\sigma}\right]$$

$$\sim \left[\alpha^{\frac{1}{1-\sigma}} + (1-\alpha)^{\frac{1}{1-\sigma}}\right]^{1-\sigma} cSL^{-1}b^{-\sigma-1}.$$

Aggregating over the b poor gives

$$b(U^{B,\mathrm{SOE}} - U^{B,\mathrm{LF}}) \sim \left[\alpha^{\frac{1}{1-\sigma}} + (1-\alpha)^{\frac{1}{1-\sigma}}\right]^{1-\sigma} cSL^{-1}\, b^{-\sigma} > 0.$$

Let

$$K := \left[\alpha^{\frac{1}{1-\sigma}} + (1-\alpha)^{\frac{1}{1-\sigma}}\right]^{1-\sigma}.$$

The corresponding welfare differences for traders is:

$$\Delta U^{C} := U^{C,\mathrm{SOE}} - U^{C,\mathrm{LF}} = \frac{K}{\sigma}\Big[L^{\sigma} p_{\mathrm{SOE}}^{-\sigma} - (L+S)^{\sigma} p_{\mathrm{LF}}^{-\sigma}\Big].$$

$$\Delta U^{C} = \frac{K}{\sigma}\Big[L^{\sigma}\big(p_{\mathrm{SOE}}^{-\sigma} - p_{\mathrm{LF}}^{-\sigma}\big) \;+\; p_{\mathrm{LF}}^{-\sigma}\big(L^{\sigma} - (L+S)^{\sigma}\big)\Big].$$

[22] under our parametric restriction $p^{LF} > p^{SOE}\ \forall b > 0$.

For large b, we get

$$\Delta U^C \;\sim\; \frac{K}{\sigma}\, p_{\mathrm{LF}}^{-\sigma}\big[\, L^\sigma - (L+S)^\sigma \,\big] \;\sim\; \frac{K}{\sigma}\, b^{-\sigma} L^{-\sigma}\big[\, L^\sigma - (L+S)^\sigma \,\big].$$

$$\Delta U^C \;\sim\; \frac{K}{\sigma}\, b^{-\sigma}\Big[\, 1 - (1+S/L)^\sigma \,\Big].$$

For $S(b) \to \infty$, we can thus say the poor agents' gain dominates, implying

$$W_b^{\mathrm{SOE}} - W_b^{\mathrm{LF}} > 0 \qquad \text{for all sufficiently large } b.$$

(ii) Now consider the case $\frac{L+S}{p} > (p^{LF})^{\phi-1}$. A very similar argument to the one above suffices.

One can similarly show that welfare differences for rich traders and the landlord are of strictly smaller order than that of the poor.

Region III: caps bind for poor and rich. In the case where the rationing caps on essential goods are binding for both the rich and the poor, an increase in the cap is Pareto improving. Hence we only consider the case where the poor are constrained in exactly one essential good. As before, we are in an economy with b large enough such that all income of the poor is spent on foodgrains. Let μ be an expenditure cap associated with any quantity cap q, so that $\mu = pq$. Without loss of generality assume $0.5 < \alpha \le 1$. The region in which the poor are constrained in exactly one good is

$$\frac{(1-\alpha)^{1/(1-\sigma)}}{\alpha^{1/(1-\sigma)} + (1-\alpha)^{1/(1-\sigma)}} L < \mu < \frac{\alpha^{1/(1-\sigma)}}{\alpha^{1/(1-\sigma)} + (1-\alpha)^{1/(1-\sigma)}} L.$$

Demands in this region are

$$f_\alpha^B = \left(\frac{\mu}{p^{QR}}, \frac{L-\mu}{p^{QR}}\right), \qquad f_\alpha^C = \left(\frac{\mu}{p^{QR}}, \frac{\mu}{p^{QR}}\right).$$

The equilibrium price solves

$$p^{QR} - 2p^{QR} f_A = bL + 2c\mu,$$

where

$$f_A = \frac{1}{2}\big[(1/\alpha)^{1/(\sigma-1)} + (1/(1-\alpha))^{1/(\sigma-1)}\big]\,(p^{QR})^{\phi-1}, \qquad \phi = \frac{\sigma}{\sigma-1}.$$

Total welfare under QR equals

$$W = \frac{c}{\sigma}\left(\frac{\mu}{p^{QR}}\right)^{\sigma} + \frac{b}{\sigma}\left[\alpha\left(\frac{\mu}{p^{QR}}\right)^{\sigma} + (1-\alpha)\left(\frac{L-\mu}{p^{QR}}\right)^{\sigma}\right] + \frac{1}{2}\left[(1/\alpha)^{1/(\sigma-1)} + (1/(1-\alpha))^{1/(\sigma-1)}\right](p^{QR})^{\phi-1}.$$

Differentiating W with respect to μ yields

$$\frac{dW}{d\mu} = p^{-\sigma}A + \Big[-p^{-\sigma-1}\Theta + M(\phi-1)p^{\phi-2}\Big]\frac{2c}{1-2M\phi\, p^{\phi-1}},$$

where

$$A = c\mu^{\sigma-1} + b\big[\alpha\mu^{\sigma-1} - (1-\alpha)(L-\mu)^{\sigma-1}\big] > 0,$$

$$\Theta = c\mu^{\sigma} + b[\alpha\mu^{\sigma} + (1-\alpha)(L-\mu)^{\sigma}] > 0,$$

and

$$M = \tfrac{1}{2}\big[(1/\alpha)^{1/(\sigma-1)} + (1/(1-\alpha))^{1/(\sigma-1)}\big].$$

For large b the equilibrium price p^{QR} is large, so $1 - 2M\phi p^{\phi-1} > 0$ and the price response $dp/d\mu > 0$. In this regime the negative price–feedback term is dominated by the strictly positive direct term $p^{-\sigma}A$. Hence $\frac{dW}{d\mu} > 0$ throughout the above μ-interval. No policy in this region can therefore be optimal.

Therefore only Region II requires further analysis.

Region II: caps bind only for the rich

Lemma 4.4 (Comparison at the border of Region II and Region III)**.** *If the quantity caps are slightly loosened such that they are no longer binding to the poor but remains binding for the rich, the utilitarian welfare is negative*

Proof. Total welfare in the essential good sphere for the traders and landless under SOE can be written as $(b+c)V(p^{SOE}, p^{SOE}, L)$, as there are two essential goods with equal prices under our primitives. Under QR, the welfare at expenditure cap μ for each essential good where the landless are unconstrained as:

$$bV(p^{QR}, p^{QR}, L) + c[V(p^{QR}, p^{QR}, 2\mu) - \Delta distort]$$

where:

$$\Delta distort = V(p^{QR}, p^{QR}, 2\mu) - 2u\left(\frac{\mu}{p^{QR}}\right)$$

measures the allocative inefficiency of QR. The welfare difference can be written as;

$$b[V(p^{SOE}, p^{SOE}, L) - V(p^{QR}, p^{QR}, L)] + c[V(p^{SOE}, p^{SOE}, L) - V(p^{QR}, p^{QR}, 2\mu)] - c\Delta distort]$$

Consider a small change from L to μ, i.e., $L + \epsilon = \mu$, we get:

$$b[V(p^{SOE}, p^{SOE}, L) - V(p^{QR}, p^{QR}, L)] + c[V(p^{SOE}, p^{SOE}, L) - V(p^{QR}, p^{QR}, 2\mu)] \tag{8}$$

$$\approx (b + c)\frac{dV}{dp}(2\Delta p) - c\frac{dV}{dI}(2\mu - L) \tag{9}$$

And market clearing gives us;

$$p^{SOE} - 2p^{SOE} f_A = bL + cL$$

$$p^{QR} - 2p^{QR} f_A = bL + 2c\mu$$

For large b, i.e, $b \to \infty$, we get

$$\Delta p = p^{SOE} - p^{QR} = -\frac{c(2\mu - L)}{1 - 2(pf_A)'} \tag{10}$$

plugging in Roy's identity $\frac{dV}{dp} = -x\frac{dV}{dI}$, equation 10 in equation 8, and using the fact that there are two essential goods whose price is going up gives us :

$$\begin{aligned}
\frac{dV}{dI}\left[-(bf_B + cf_C)2\Delta p - c(2\mu - L)\right] &= \frac{dV}{dI}c(2\mu - L)\left[\frac{2(bf_B + cf_C)}{1 - 2(pf)'} - 1\right] \\
&= \frac{dV}{dI}c(2\mu - L)\left[\frac{2(1/2 - f_A)}{1 - 2(pf)'} - 1\right] \\
&= \frac{dV}{dI}c(2\mu - L)\left[\frac{2pf'}{1 - 2pf' - 2f}\right]
\end{aligned}$$

The order of this object is:

$$b^{-\sigma} p^{\frac{1}{\sigma - 1}}$$

Whereas the order of $\Delta distort$ is $b^{-\sigma}$. To see this note:

$$\Delta distort = V(p, p, 2\mu) - 2u\Big(\frac{\mu}{p}\Big) = \frac{\mu^{\sigma} p^{-\sigma}}{\sigma}\big(K2^{\sigma} - 2C_u\big) = \Theta\big(\mu^{\sigma} p^{-\sigma}\big) \sim \Theta\big(b^{-\sigma}\big)$$

where C_u is a positive constant. Hence, $\Delta distort$ is bigger for large b and the total welfare effect is negative.

Now we consider the case when caps don't bind for poor and the rich are constrained by μ but $\mu >> \frac{\alpha^{1/(1-\sigma)}}{\alpha^{1/(1-\sigma)}+(1-\alpha)^{1/(1-\sigma)}}L$. In this case, the welfare difference between the two regimes can be written as

$$W^{\text{SOE}} - W^{\text{QR}} = b\,\frac{1}{\sigma}L^\sigma \Phi(\alpha,\sigma)\big(p_{\text{SOE}}^{-\sigma} - p_{\text{QR}}^{-\sigma}\big) + c\,\frac{1}{\sigma}L^\sigma p_{\text{SOE}}^{-\sigma}\Phi(\alpha,\sigma) - c\,\frac{1}{\sigma}p_{\text{QR}}^{-\sigma}\mu^\sigma, \tag{11}$$

where

$$\Phi(\alpha,\sigma) = \Big[\alpha^{\frac{1}{1-\sigma}} + (1-\alpha)^{\frac{1}{1-\sigma}}\Big]^{1-\sigma}.$$

We now collect the terms of leading order in b, using that both prices satisfy $p_{\text{SOE}} = bL + O(1)$ and $p_{\text{QR}} = bL + O(1)$ as $b \to \infty$. A direct expansion shows that the leading contributions of order $b^{-\sigma}$ in (11) are

$$b\,L^\sigma \Phi(\alpha,\sigma)\,\frac{c(2\mu - L)}{1-2(pf_A)'}\,(bL)^{-\sigma-1} \;+\; \frac{c}{\sigma}\,(bL)^{-\sigma}\big(L^\sigma\Phi(\alpha,\sigma) - \mu^\sigma\big).$$

Using $\lim_{b\to\infty}\big(1-2(pf_A)'\big) = 1$ and simplifying, we obtain

$$W^{\text{SOE}} - W^{\text{QR}} = c\,b^{-\sigma}\left(\Phi(\alpha,\sigma)\Big(2\tfrac{\mu}{L} - 1\Big) + \frac{1}{\sigma}\Big(\Phi(\alpha,\sigma) - \big(\tfrac{\mu}{L}\big)^\sigma\Big)\right) + o\big(b^{-\sigma}\big).$$

For ease of exposition normalize $L = 1$, hence we are now in the region such that $\mu > \frac{\alpha^{1/(1-\sigma)}}{\alpha^{1/(1-\sigma)}+(1-\alpha)^{1/(1-\sigma)}} > \frac{1}{2}$. and $\frac{\mu}{L} = \mu$. $\mu > 0$ is now understood in units of L and in particular $\mu > \frac{1}{2}$ means $\frac{\mu}{L} > \frac{1}{2}$. The leading-order coefficient of $b^{-\sigma}$ is then

$$c\,b^{-\sigma}\left(\Phi(\alpha,\sigma)(2\mu - 1) + \frac{1}{\sigma}\big(\Phi(\alpha,\sigma) - \mu^\sigma\big)\right).$$

We now write a lemma to show this is positive.

Lemma 4.5. *Let $\sigma < 0$ and $\alpha \in \big(\frac{1}{2}, 1\big)$. Define*

$$\Phi(\alpha,\sigma) = \Big[\alpha^{\frac{1}{1-\sigma}} + (1-\alpha)^{\frac{1}{1-\sigma}}\Big]^{1-\sigma},$$

and for $\mu > 0$ set

$$S(\mu) := \Phi(\alpha,\sigma)(2\mu - 1) + \frac{1}{\sigma}\big(\Phi(\alpha,\sigma) - \mu^\sigma\big).$$

Then for all $\mu > \frac{1}{2}$ one has

$$S(\mu) > 0.$$

Proof. For brevity write $\Phi := \Phi(\alpha, \sigma) > 0$. Consider $S(\mu)$ as a function of $\mu > 0$:

$$S(\mu) = \Phi(2\mu - 1) + \frac{1}{\sigma}(\Phi - \mu^{\sigma}).$$

Step 1: Convexity and the unique minimiser. Compute the first and second derivatives with respect to μ:

$$S'(\mu) = 2\Phi + \frac{1}{\sigma}(-\sigma\mu^{\sigma-1}) = 2\Phi - \mu^{\sigma-1},$$
$$S''(\mu) = -(\sigma - 1)\mu^{\sigma-2}.$$

Since $\sigma < 0$ implies $\sigma - 1 < 0$, it follows that $-(\sigma - 1) > 0$, and hence $S''(\mu) > 0$ for all $\mu > 0$. Therefore S is strictly convex.

The unique critical point satisfies $S'(\mu) = 0$, that is,

$$2\Phi = \mu^{\sigma-1} \quad \Longrightarrow \quad \mu_* = (2\Phi)^{\frac{1}{\sigma-1}}.$$

Because S is strictly convex, μ_* is the unique global minimiser of S.

At $\mu = \mu_*$ we have

$$\mu_*^{\sigma-1} = 2\Phi \quad \Longrightarrow \quad \mu_*^{\sigma} = \mu_*^{\sigma-1}\mu_* = 2\Phi\,\mu_*.$$

Substituting this into $S(\mu)$ yields

$$\begin{aligned} S(\mu_*) &= \Phi(2\mu_* - 1) + \frac{1}{\sigma}(\Phi - \mu_*^{\sigma}) \\ &= \Phi(2\mu_* - 1) + \frac{1}{\sigma}(\Phi - 2\Phi\mu_*) \\ &= \Phi\Big[(2\mu_* - 1) + \frac{1}{\sigma}(1 - 2\mu_*)\Big] \\ &= \Phi\,(2\mu_* - 1)\Big(1 - \frac{1}{\sigma}\Big). \end{aligned}$$

Step 2: A bound on $\Phi(\alpha, \sigma)$ and the location of μ_*. Define

$$f(x) := x^{\frac{1}{1-\sigma}}, \qquad x \in (0, 1).$$

Since $1 - \sigma > 1$ for $\sigma < 0$, the exponent $\frac{1}{1-\sigma} \in (0, 1)$, and f is strictly concave on $(0, 1)$. Hence the function

$$g(\alpha) := f(\alpha) + f(1 - \alpha) = \alpha^{\frac{1}{1-\sigma}} + (1 - \alpha)^{\frac{1}{1-\sigma}}$$

is strictly maximised at $\alpha = \frac{1}{2}$, and for $\alpha \neq \frac{1}{2}$ one has

$$g(\alpha) < 2\Big(\frac{1}{2}\Big)^{\frac{1}{1-\sigma}}.$$

Raising both sides to the power $1-\sigma > 0$ gives, for $\alpha > \frac{1}{2}$,

$$\Phi(\alpha,\sigma) = \big[g(\alpha)\big]^{1-\sigma} < \Big[2\Big(\frac{1}{2}\Big)^{\frac{1}{1-\sigma}}\Big]^{1-\sigma} = 2^{-\sigma}.$$

Thus, under $\alpha > \frac{1}{2}$,

$$\Phi(\alpha,\sigma) < 2^{-\sigma}. \tag{12}$$

We now compare μ_* with $\frac{1}{2}$. Using (12),

$$\mu_* = (2\Phi)^{\frac{1}{\sigma-1}}.$$

Since $\sigma - 1 < 0$, the map $x \mapsto x^{1/(\sigma-1)}$ is strictly decreasing on $(0,\infty)$. From $\Phi < 2^{-\sigma}$ it follows that

$$2\Phi < 2^{1-\sigma} \quad \Longrightarrow \quad (2\Phi)^{\frac{1}{\sigma-1}} > \big(2^{1-\sigma}\big)^{\frac{1}{\sigma-1}} = \frac{1}{2}.$$

Hence

$$\mu_* > \frac{1}{2}, \qquad \text{so that} \qquad 2\mu_* - 1 > 0.$$

Step 3: Sign of the minimum and of $S(\mu)$ for $\mu > \frac{1}{2}$. Recall

$$S(\mu_*) = \Phi\Big(1 - \frac{1}{\sigma}\Big)(2\mu_* - 1).$$

We have $\Phi > 0$, $2\mu_* - 1 > 0$, and since $\sigma < 0$, also

$$1 - \frac{1}{\sigma} > 0.$$

Therefore $S(\mu_*) > 0$.

Because S is strictly convex and μ_* is its unique global minimiser, we obtain

$$S(\mu) \geq S(\mu_*) > 0 \qquad \text{for all } \mu > 0.$$

In particular, for all $\mu > \frac{1}{2}$,

$$S(\mu) > 0,$$

which is the desired claim. □

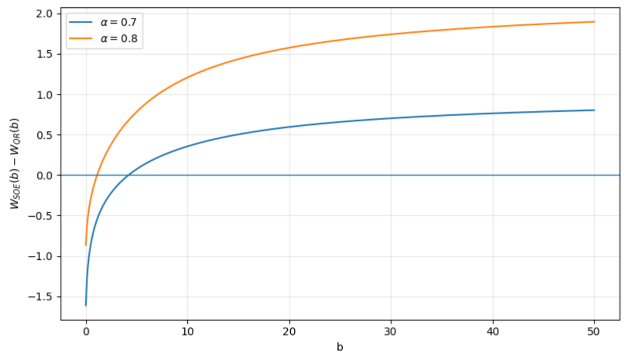

Figure 2: Welfare Difference between SOE and optimal QR policy on the y-axis and the number of poor b on the y-axis. Utility functions for two types of agent: $\alpha k ln f_1 + (1-\alpha) k ln f_2 + l + s$ and $(1-\alpha) k ln f_1 + \alpha k ln f_2 + l + s$ where $k = 4.5$. Endowment of leisure for everyone is $L = 1$. Relative number of rich are $c = 1.5$. Relative number of landowners $a = 1$.

□

A.3 Equilibrium Analysis for the Two-Luxury-Good Economy

This appendix provides the formal derivations for the model in Section 3.2, which features two luxury goods. The primitives–preferences, endowments, and agent types–are as specified in the main text.

Endowment Type	Foodgrain	Time	Lapis Lazuli	Silver
Landowner	1	L	0	0
Landless Laborer	0	L	0	0
Trader	0	L	S	S

Table 6: Endowment Distribution

Equilibrium under Laissez-Faire. The laissez-faire equilibrium is analogous to the baseline model. With quasi-linear preferences, the price of leisure is normalized to 1. With quasi-linear preferences and marginal utilities for luxuries in $\{1/4, 1/2\}$, the only competitive equilibrium supporting positive holdings is $p_{s_1} = p_{s_2} = 1/2$. If $p_s > 1/2$, no agent is willing to hold any luxury; if $p_s < 1/2$, both trader types demand strictly more of their favored luxury than the aggregate endowment. Traders use their full wealth, including luxury endowments, to demand essentials. The market-clearing condition for the essential good is:

$$a\frac{k}{p_f} + b\frac{L}{p_f} + c\frac{k}{p_f} = 1,$$

where $a = 1$ is the mass of landowners. Solving for the price and substituting back into demand functions yields the equilibrium consumption levels presented in Table 7.

	Landowner	**Landless**	**Trader**
Foodgrain Consumption	$\frac{k}{b \cdot L + (1+c) \cdot k}$	$\frac{L}{b \cdot L + (1+c) \cdot k}$	$\frac{k}{b \cdot L + (1+c) \cdot k}$

Table 7: Foodgrain Consumption under Laissez-Faire

Equilibrium under Spheres of Exchange. Under SOE, the budget constraints are segregated. In the essential-goods sphere, the constraint is $p_f(f - \hat{f}) + p_l(l - \hat{L}) \leq 0$. In the luxury-goods sphere, it is $p_{s_1}(S_1 - \hat{S}_1) + p_{s_2}(S_2 - \hat{S}_2) \leq 0$.

Luxury-Sphere Equilibrium: Within the luxury sphere, traders have symmetric endowments (S, S) but heterogeneous linear utilities. The equilibrium is characterized by equal prices, $p_{s_1} = p_{s_2} = p_s$. Given the specified marginal utilities $(1/4, 1/2)$ and $(1/2, 1/4)$, market clearing implies $1/4 \leq p_s \leq 1/2$. At these price, all luxury goods are traded efficiently: Type 1 traders sell s_1 to buy s_2, and Type 2 traders do the reverse. This intra-sphere trade realizes all potential gains without distortion.

Essential-Sphere Equilibrium: Crucially, traders cannot transfer wealth from the luxury sphere. Thus, their income for purchasing the essential good is solely L, identical to that of landless laborers. Landowners, as net sellers, have income $p_f + L$. The essential-good market clears when:

$$\frac{k}{p_f^{\text{SOE}}} + (b+c)\frac{L}{p_f^{\text{SOE}}} = 1.$$

Solving for the price gives $p_f^{\text{SOE}} = k + (b+c)L$. Substituting into the demand functions yields the consumption allocations in Table 8.

The change in consumption for the poor (landless laborers) relative to laissez-faire is:

$$f_B^{\text{SOE}} - f_B^{\text{LF}} = \frac{L}{k + (b+c)L} - \frac{L}{k(1+c) + bL} = \frac{cL(k-L)}{p_f^{\text{SOE}} \cdot p_f^{\text{LF}}} > 0.$$

This positive "Consumption Effect" demonstrates the redistributive benefit of the SOE regime. By restricting traders' demand, SOE lowers the price of essentials, making them

	Landowner	**Landless**	**Trader**
Foodgrain Consumption	$\dfrac{k}{(b+c)\cdot L+k}$	$\dfrac{L}{(b+c)\cdot L+k}$	$\dfrac{L}{(b+c)\cdot L+k}$

Table 8: Foodgrain Consumption under Spheres of Exchange

more affordable for the poor.

Non-Distortionary Taxation: $\tau \leq 1$

Claim 4.3. *If $\tau \leq 1$, luxury–luxury trade remains efficient. The maximum tax rate consistent with full trade is $\tau = 1$, yielding total revenue $R^{\text{ND}} = (c \cdot S)/4$.*

Proof. WLOG, consider one particular type of trader (Call Type 1). They are endowed with (S, S) units of (s_1, s_2) but value s_2 more highly ($MU_{s_2} = 1/2$) than s_1 ($MU_{s_1} = 1/4$). They will want to sell some s_1 to buy more s_2.

Let p_s be the seller's price (the price received by the seller). Due to the tax τ, a buyer must pay $(1+\tau)p_s$ per unit. For trade to occur, the buyer's price cannot exceed the marginal utility of the good for the buyer, which is 1/2. In equilibrium, the buyer's price equals this marginal utility:

$$(1+\tau)p_s = \frac{1}{2} \quad \Rightarrow \quad p_s = \frac{1}{2(1+\tau)}.$$

A Type 1 trader will sell one unit of s_1 if the utility from the additional s_2 they can buy exceeds the utility lost from selling s_1:

$$\left(\frac{p_s}{(1+\tau)p_s}\right) \cdot \frac{1}{2} > \frac{1}{4}.$$

The term $p_s/[(1+\tau)p_s] = 1/(1+\tau)$ is the number of units of s_2 that can be bought with the revenue from selling one unit of s_1. Multiplying by the marginal utility of s_2 (1/2) gives the total utility gain. The right side is the marginal utility of s_1 (1/4).

Simplifying the inequality:

$$\frac{1}{1+\tau} \cdot \frac{1}{2} > \frac{1}{4} \quad \Longleftrightarrow \quad \frac{1}{1+\tau} > \frac{1}{2} \quad \Longleftrightarrow \quad 2 > 1+\tau \quad \Longleftrightarrow \quad \tau < 1.$$

Similarly, a Type 2 trader (with $MU_{s_1} = 1/2$, $MU_{s_2} = 1/4$) will want to sell s_2 to buy s_1 under the same condition, $\tau < 1$. When $\tau = 1$, traders are indifferent, so we include it in the efficient trade regime.

Thus, for $\tau \leq 1$, all S units of the less-valued good are sold by each trader, and the entire luxury endowment is efficiently reallocated. The equilibrium seller's price is $p_s = 1/(2(1+\tau))$.

Tax revenue is collected on each unit traded. Since each of the c traders sells S units, the total quantity traded is $c \cdot S$. The revenue per unit traded is $\tau \cdot p_s = \tau \cdot \frac{1}{2(1+\tau)}$. Therefore, total revenue is:

$$R^{\mathrm{ND}}(\tau) = \left(\tau \cdot \frac{1}{2(1+\tau)}\right) \cdot (c \cdot S) = \frac{c}{2} \cdot \frac{\tau}{1+\tau} \cdot S.$$

This revenue function is maximized at $\tau = 1$, yielding $R^{\mathrm{ND}}(\tau = 1) = \frac{c}{2} \cdot \frac{1}{2} \cdot S = \frac{c \cdot S}{4}$. □

Distortionary Taxation: $\tau > 1$

Claim 4.4. *If $\tau > 1$, luxury–luxury trade collapses. Traders only sell luxuries to finance essential consumption. The revenue-maximizing tax is $\tau^* = \sqrt{\frac{2k}{L}} - 1$, yielding maximal revenue $R^{\mathrm{D}}(\tau^*) = c \cdot \left(\sqrt{2k} - \sqrt{L}\right)^2$, which is independent of S.*

Proof. When $\tau > 1$, the tax wedge eliminates gains from trade within the luxury sphere. Traders no longer exchange s_1 for s_2. Instead, they hold their luxury endowments and may sell them solely to purchase the essential good.

In this regime, the demand for luxuries comes only from landowners, whose marginal utility for either luxury is 1/2. Thus, the buyer's price for luxuries is $p_s^{\text{buyer}} = 1/2$. The seller's price is therefore $p_s = p_s^{\text{buyer}}/(1+\tau) = 1/(2(1+\tau))$.

A trader will sell luxuries to purchase essentials if their marginal utility from essential consumption exceeds the marginal utility from holding the luxury. Given the quasi-linear structure, this leads to an interior choice. The trader's optimization problem, considering they can sell luxuries at price $p_s = 1/(2(1+\tau))$ to earn income for essentials, yields the demand function:

$$f = \max\left\{\frac{2k}{p_f(1+\tau)}, \frac{L}{p_f}\right\}. \tag{13}$$

This reflects that traders first use their labor income L and then sell luxuries if their subsistence demand k is not yet met.

Tax revenue is generated only from the sale of luxuries to fund essential consumption beyond L. From the demand function, if $f > L/p_f$, then the amount spent on essentials beyond labor income is $p_f f - L$. Therefore, revenue per trader is:

$$R^{\mathrm{D}}_{\text{per-trader}}(\tau) = \tau \cdot (p_f f - L) = \tau \cdot \left(\frac{2k}{1+\tau} - L\right),$$

where we substitute the optimal $f = 2k/(p_f(1+\tau))$ from (13). Aggregating over all c traders gives the total revenue $R^{\mathrm{D}}(\tau) = c \cdot \tau\left(\frac{2k}{1+\tau} - L\right)$.

To find the revenue-maximizing tax rate τ^*, we maximize $R^{\mathrm{D}}(\tau)$ for $\tau > 1$:

$$\frac{d}{d\tau} R^{\mathrm{D}}(\tau) = c \cdot \left(\frac{2k}{1+\tau} - L - \frac{2k\tau}{(1+\tau)^2} \right) = 0.$$

Multiplying through by $(1+\tau)^2$:

$$2k(1+\tau) - L(1+\tau)^2 - 2k\tau = 0 \quad \Longleftrightarrow \quad 2k - L(1+\tau)^2 = 0.$$

Solving for τ:

$$(1+\tau)^2 = \frac{2k}{L} \quad \Longleftrightarrow \quad \tau^* = \sqrt{\frac{2k}{L}} - 1.$$

The second derivative is negative, confirming a maximum. Substituting τ^* back into the revenue function yields the maximal revenue:

$$R^{\mathrm{D}}(\tau^*) = c \cdot \left(\sqrt{\frac{2k}{L}} - 1 \right) \left(\frac{2k}{\sqrt{\frac{2k}{L}}} - L \right) = c \cdot \left(\sqrt{\frac{2k}{L}} - 1 \right) \left(\sqrt{2kL} - L \right).$$

Factoring gives:

$$R^{\mathrm{D}}(\tau^*) = c \cdot \left(\sqrt{2k} - \sqrt{L} \right)^2.$$

This expression is independent of the luxury endowment S, as it is determined solely by the parameters of essential good demand (k) and labor endowment (L). □

A.3.1 Proof of Lemma 1.1: Subsidies vs Direct Transfers

We divide the comparison of welfare into two cases one where $\tau \leq 1$ and one where $\tau > 1$.

Direct Transfers (DT) under Non-Distortionary Taxation

Let the maximum tax revenue under non-distortionary (ND) taxation be R_{ND}.Under ND taxation, each poor agent receives R^{ND}/b. Let p^{ND} be the resulting equilibrium price. Demands satisfy

$$f_B = \frac{L + \frac{R^{ND}}{b}}{p^{ND}}, \qquad f_A = f_C = \frac{k}{p^{ND}},$$

and market clearing implies

$$p^{ND} = (1+c)k + bL + R^{ND} = p^{LF} + R^{ND}.$$

Thus,

$$f_B^{ND} = f_B^{LF} + \frac{R^{ND}}{b} \frac{(1+c)k}{p^{ND} p^{LF}}.$$

Poor consumption strictly increases relative to laissez-faire.

Subsidies equilibrium under non-distortionary taxation

Under SB, the planner uses R to subsidize essential-good purchases at rate τ^{SB}. If p^{SB} is the consumer price, sellers receive $(1+\tau^{SB})p^{SB}$. Budget balance requires:

$$p^{SB}\tau^{SB}(bf_B + cf_C) = R^{ND}.$$

Using $f_B = L/p^{SB}$ and $f_C = k/p^{SB}$, market clearing yields

$$p^{SB} = \Big(\frac{1}{1+\tau^{SB}} + c\Big)k + bL = p^{LF} - \left(1 - \frac{1}{1+\tau^{SB}}\right)k.$$

Thus the poor consume

$$f_B^{SB} = f_B^{LF} + \frac{R^{ND}L}{(bL+ck+R^{ND})} \cdot \frac{k}{p^{SB}p^{LF}}.$$

Subsidies increase poor consumption but also raise traders' demand by lowering price, partially offsetting the intended effect.

Comparison of DT vs SB under Non-Distortionary taxation. We show that for any fixed non-distortionary revenue R^{ND}, direct transfers (DT) yield strictly higher consumption for the poor than subsidies (SB) for all $b > 0$ and $c > 0$. The argument proceeds by deriving an exact closed-form expression for

$$\Delta f \equiv f_B^{\text{DT}} - f_B^{\text{SB}},$$

Step 1: Consumption under DT and SB. Under direct transfers, each poor agent receives R^{ND}/b, so

$$f_B^{\text{DT}} = \frac{L + R^{\text{ND}}/b}{p^{\text{DT}}}, \qquad p^{\text{DT}} = (1+c)k + bL + R^{\text{ND}}.$$

Under subsidies, the planner chooses τ^{SB} to satisfy the non-distortionary revenue requirement:

$$p^{\text{SB}}\tau^{\text{SB}}(bf_B + cf_C) = R^{\text{ND}} \quad \Rightarrow \quad \tau^{\text{SB}} = \frac{R^{\text{ND}}}{bL+ck}.$$

With $f_B = L/p^{\text{SB}}$ and $f_C = k/p^{\text{SB}}$, the equilibrium price is

$$p^{\text{SB}} = \left(\frac{1}{1+\tau^{\text{SB}}} + c\right)k + bL = \frac{bL+ck}{bL+ck+R^{\text{ND}}}k + ck + bL.$$

Poor consumption under SB is therefore $f_B^{\text{SB}} = L/p^{\text{SB}}$.

Step 2: Exact expression for Δf. Combining the above expressions and simplifying yields

$$\Delta f = \frac{L + R^{\mathrm{ND}}/b}{p^{\mathrm{DT}}} - \frac{L}{p^{\mathrm{SB}}}.$$

A direct algebraic manipulation (expanding both fractions over the common denominator $p^{\mathrm{DT}}p^{\mathrm{SB}}$ and substituting the expression for τ^{SB}) gives the exact closed form

$$\Delta f = \frac{R^{\mathrm{ND}}\, c\, k}{b\,(bL + ck)\,\big((1+c)k + bL + R^{\mathrm{ND}}\big)}. \tag{14}$$

Every term in (14) is strictly positive, implying

$$\Delta f > 0 \qquad \text{for all } b > 0.$$

Conclusion. Direct transfers yield strictly greater consumption for the poor than subsidies for any admissible b in the non-distortionary case. □

Distortionary Taxation ($\tau > 1$): Proof Under distortionary taxation, luxury–luxury trade collapses and traders only sell luxuries to finance essential consumption. Their demand is

$$f_C = \frac{2k}{p_f(1+\tau)}.$$

Let R^{D} denote the revenue under the optimal distortionary tax $\tau^* > 1$.

Under direct transfers (DT), the market-clearing condition is

$$\frac{k}{p^{\mathrm{DT}}} + b \cdot \frac{L + R^{\mathrm{D}}/b}{p^{\mathrm{DT}}} + c \cdot \frac{2k}{p^{\mathrm{DT}}(1+\tau)} = 1,$$

which implies

$$p^{\mathrm{DT}} = \left(1 + \frac{2c}{1+\tau}\right) k + bL + R^{\mathrm{D}}, \qquad f_B^{\mathrm{DT}} = \frac{L + R^{\mathrm{D}}/b}{p^{\mathrm{DT}}}.$$

Under subsidies (SB), the planner's budget constraint is

$$p^{\mathrm{SB}}\tau^{\mathrm{SB}} \left(b \cdot \frac{L}{p^{\mathrm{SB}}} + c \cdot \frac{2k}{p^{\mathrm{SB}}(1+\tau)} \right) = R^{\mathrm{D}},$$

so

$$\tau^{\mathrm{SB}} = \frac{R^{\mathrm{D}}}{bL + \frac{2ck}{1+\tau}}.$$

The market-clearing condition is

$$\frac{k}{p^{\mathrm{SB}}(1+\tau^{\mathrm{SB}})} + b \cdot \frac{L}{p^{\mathrm{SB}}} + c \cdot \frac{2k}{p^{\mathrm{SB}}(1+\tau)} = 1,$$

which yields

$$p^{\mathrm{SB}} = \frac{k}{1+\tau^{\mathrm{SB}}} + bL + \frac{2ck}{1+\tau}, \qquad f_B^{\mathrm{SB}} = \frac{L}{p^{\mathrm{SB}}}.$$

Define $A \equiv bL + \frac{2ck}{1+\tau}$. Substituting $\tau^{\mathrm{SB}} = R^{\mathrm{D}}/A$ and simplifying the difference

$$\Delta f \equiv f_B^{\mathrm{DT}} - f_B^{\mathrm{SB}} = \frac{L + R^{\mathrm{D}}/b}{p^{\mathrm{DT}}} - \frac{L}{p^{\mathrm{SB}}},$$

we obtain the exact expression

$$\Delta f = \frac{2ckR^{\mathrm{D}}}{(1+\tau)\, b\, A\, p^{\mathrm{DT}}},$$

with

$$A = bL + \frac{2ck}{1+\tau}, \qquad p^{\mathrm{DT}} = \left(1 + \frac{2c}{1+\tau}\right) k + bL + R^{\mathrm{D}}.$$

Since all terms in the denominator are strictly positive, it follows that $\Delta f > 0$ for all $b > 0$.

Thus, under distortionary taxation as well, direct transfers yield strictly higher consumption for the poor than subsidies for all b.

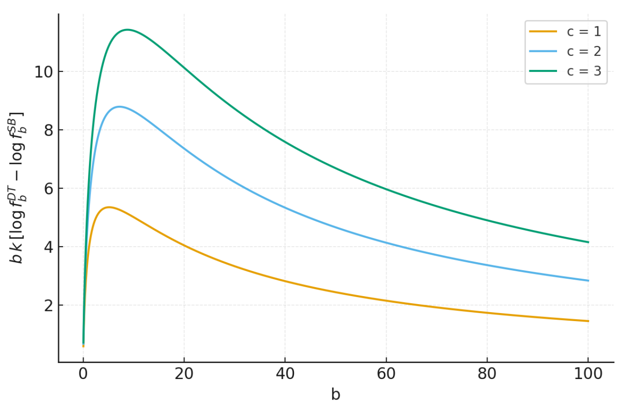

Figure 3: Welfare difference (ΔW) between subsidies and direct transfers under non-distortionary taxation on the y-axis and relative number of poor b on the x-axis. Utility functions for two types of agent: $klnf + l + \gamma_1 s_1 + \gamma_2 s_2$ and $klnf + l + \gamma_2 s_1 + \gamma_1 s_2$ where $k = 4.5, \gamma_1 = 1/2, \gamma_2 = 1/4.$. Endowment of leisure for everyone is $L = 1$. Relative number of landowners $a = 1$.

	Landowner	**Landless**	S_1 **Trader**	S_2 **Trader**
Foodgrain Consumption	$\frac{k}{p_{DT}}$	$\frac{L+\frac{Sc}{4b}}{p_{DT}}$	$\frac{k}{p_{DT}}$	$\frac{k}{p_{DT}}$

Table 9: Foodgrain Consumption by Group

A.3.2 Proof of Lemma 1.2

We compare total welfare under the DT and SB regimes via

$$\frac{W_{DT} - W_{SB}}{k} = b \log \frac{f_B^{DT}}{f_B^{SB}} + c \log \frac{f_C^{DT}}{f_C^{SB}} + \log \frac{f_A^{DT}}{f_A^{SB}}.$$

Using the expressions for $f_A^{DT}, f_B^{DT}, f_C^{DT}$ and $f_A^{SB}, f_B^{SB}, f_C^{SB}$, this can be written as

$$\frac{W_{DT} - W_{SB}}{k} = b \log \left(1 + \frac{\Delta f_B}{f_B^{SB}}\right) + c \log \left(\frac{p_{SB}}{p_{DT}}\right) + \log \left(\frac{p_{SB}(1+\tau)}{p_{DT}}\right),$$

where $\Delta f_B := f_B^{DT} - f_B^{SB}$.

Suppose now that $b \to \infty$ and that, from the equilibrium expressions derived above,

$$1 + \frac{\Delta f_B}{f_B^{SB}} = 1 + O\big(b^{-2}\big), \qquad \frac{p_{SB}}{p_{DT}} = 1 + O\big(b^{-1}\big), \qquad \frac{p_{SB}(1+\tau)}{p_{DT}} = 1 + O\big(b^{-1}\big),$$

as $b \to \infty$.

Using the standard expansion

$$\log(1+x) = x + O(x^2) \quad \text{as } x \to 0,$$

it follows that

$$\log \left(1 + \frac{\Delta f_B}{f_B^{SB}}\right) = O\big(b^{-2}\big), \qquad \log \left(\frac{p_{SB}}{p_{DT}}\right) = O\big(b^{-1}\big), \qquad \log \left(\frac{p_{SB}(1+\tau)}{p_{DT}}\right) = O\big(b^{-1}\big),$$

as $b \to \infty$.

Hence

$$\frac{W_{DT} - W_{SB}}{k} = b \cdot O\big(b^{-2}\big) + c \cdot O\big(b^{-1}\big) + O\big(b^{-1}\big) = O\big(b^{-1}\big) \xrightarrow[b\to\infty]{} 0.$$

Therefore the difference in total welfare between DT and SB, normalized by k, converges to zero in the limit considered.

A.3.3 Proof of Proposition 4

This appendix proves Proposition 4 by comparing welfare under spheres of exchange (SOE) and direct transfers (DT) in the non-distortionary and distortionary tax regimes.

Non-Distortionary Taxation ($\tau \leq 1$)

Claim 4.5. *Let $k, L > 0$ and $c > 0$ be fixed, with $k > L$. Under non-distortionary taxation at $\tau = 1$, suppose the luxury endowment $S(b)$ satisfies $\lim_{b\to\infty} S(b)/b = 0$. Then, for sufficiently large b, $W^{\text{SOE}} > W^{\text{DT}}$.*

Proof. At $\tau = 1$, total non-distortionary revenue is $R^{\text{ND}} = \frac{c}{4}S(b)$. The equilibrium prices are

$$p^{\text{SOE}} = k + (b+c)L, \qquad p^{\text{DT}} = (1+c)k + bL + \frac{c}{4}S(b).$$

The welfare difference can be written as

$$\begin{aligned}
\Delta_W &\equiv W^{\text{SOE}} - W^{\text{DT}} \\
&= k \log \frac{p^{\text{DT}}}{p^{\text{SOE}}} + (b+c)k \log \frac{p^{\text{DT}}}{p^{\text{SOE}}} + bk \left[\log \frac{L}{p^{\text{SOE}}} - \log \frac{L + \frac{c}{4}\frac{S(b)}{b}}{p^{\text{DT}}} \right] \\
&= (b+c+1)k \log \frac{p^{\text{DT}}}{p^{\text{SOE}}} + bk \log \frac{L p^{\text{DT}}}{\left(L + \frac{c}{4}\frac{S(b)}{b}\right) p^{\text{SOE}}}.
\end{aligned}$$

Define

$$p^{\text{SOE}} = bL + A, \qquad p^{\text{DT}} = bL + B(b),$$

where

$$A \equiv k + cL, \qquad B(b) \equiv (1+c)k + \frac{c}{4}S(b).$$

Then

$$\frac{p^{\text{DT}}}{p^{\text{SOE}}} = \frac{bL + B(b)}{bL + A} = 1 + \frac{B(b) - A}{bL + A}.$$

Since $S(b)/b \to 0$, we have $B(b) - A = c(k-L) + \frac{c}{4}S(b)$ with $\frac{S(b)}{b} \to 0$, and thus

$$\frac{B(b) - A}{bL + A} = \frac{c(k-L)}{bL} + \frac{c}{4}\frac{S(b)}{bL} + o\Big(\frac{1}{b}\Big) \quad \text{as } b \to \infty.$$

Using $\log(1+x) = x + o(x)$ as $x \to 0$, it follows that

$$\log \frac{p^{\text{DT}}}{p^{\text{SOE}}} = \frac{c(k-L)}{bL} + \frac{c}{4}\frac{S(b)}{bL} + o\Big(\frac{1}{b}\Big). \tag{15}$$

For the second logarithm, observe that

$$\log \frac{Lp^{\mathrm{DT}}}{\left(L+\frac{c}{4}\frac{S(b)}{b}\right)p^{\mathrm{SOE}}} = \log \frac{p^{\mathrm{DT}}}{p^{\mathrm{SOE}}} + \log \frac{L}{L+\frac{c}{4}\frac{S(b)}{b}} = \log \frac{p^{\mathrm{DT}}}{p^{\mathrm{SOE}}} - \log\left(1+\frac{c}{4}\frac{S(b)}{bL}\right).$$

Using again $\log(1+x) = x + o(x)$ and (15),

$$\log \frac{L}{L+\frac{c}{4}\frac{S(b)}{b}} = \frac{c}{4}\frac{S(b)}{bL} + o\left(\frac{1}{b}\right),$$

so the $S(b)$-terms cancel and we obtain

$$\log \frac{Lp^{\mathrm{DT}}}{\left(L+\frac{c}{4}\frac{S(b)}{b}\right)p^{\mathrm{SOE}}} = \frac{c(k-L)}{bL} + o\left(\frac{1}{b}\right). \tag{16}$$

Substituting (15) and (16) into Δ_W yields

$$\begin{aligned}\Delta_W &= (b+c+1)k\left[\frac{c(k-L)}{bL} + \frac{c}{4}\frac{S(b)}{bL} + o\left(\frac{1}{b}\right)\right] + bk\left[\frac{c(k-L)}{bL} + o\left(\frac{1}{b}\right)\right] \\ &= \frac{kc}{L}\left[(k-L) + \frac{S(b)}{4}\right] + \frac{kc}{L}(k-L) + o(1) \\ &= \frac{2kc}{L}(k-L) + \frac{kc}{4L}S(b) + o(1) \quad \text{as } b \to \infty.\end{aligned}$$

Since $k > L$ and $c > 0$, we have $\frac{2kc}{L}(k-L) > 0$, and the term $\frac{kc}{4L}S(b)$ is non-negative. Hence

$$\liminf_{b\to\infty} \Delta_W > 0,$$

which implies that $W^{\mathrm{SOE}} > W^{\mathrm{DT}}$ for all sufficiently large b.[23] □

Distortionary Taxation ($\tau > 1$)

Lemma 4.6. *Under distortionary taxation, the welfare gain to the landless from redistribution satisfies*

$$W^{\mathrm{DT}} - W^{\mathrm{SOE}} \leq \frac{R^{\mathrm{D}}k^2}{L^2} \cdot \frac{1}{b+c}.$$

Proof. Under distortionary taxation the revenue–maximising tax rate is $\tau^* = \sqrt{\frac{2k}{L}} - 1$ (see

[23] We use $\liminf \Delta_W$ to not have to worry about the existence of the limit.

Appendix A.3), which generates maximal revenue

$$R^{\mathrm{D}} \;=\; c\big(\sqrt{2k}-\sqrt{L}\big)^2.$$

Because all agents are endowed with baseline L, the rich traders income is always greater than or equal to the poor. Given food is a normal good, the rich always consume greater than or equal to the poor. Thus the *maximum* feasible poor consumption, consistent with market clearing, is obtained in the auxiliary allocation where poor and traders consume the same amount, conditional on maximum revenue being raised.[24]

$$f_B^{(2)} = f_C^{(2)} = \frac{L+\frac{R^D}{b+c}}{p^{(2)}}, \qquad p^{(2)} = bL + R^D + k + cL.$$

We can write that as:

Poor consumption under SOE is

$$f_B^{SOE} = \frac{L}{p^{SOE}}, \qquad p^{SOE} = bL + (1+c)k.$$

Hence the welfare gain attributable to redistribution is bounded by

$$\Phi(b) := bk\left[\log f_B^{DT} - \log f_B^{SOE}\right] \le bk\left[\log\left(1+\frac{R^D}{(b+c)L}\right) - \log\left(1+\frac{R^D}{(b+c)L+k}\right)\right].$$

For $x \ge y \ge 0$, the function $\log(1+x)-\log(1+y)$ is increasing in x, decreasing in y,

[24] One can quickly check this claim. Define

$$\Delta := \sqrt{2kL} - L > 0 \qquad (\text{since } 2k > L).$$

In the distortionary equilibrium where maximum revenue is generated (Case 1): $\tau = \sqrt{\frac{2k}{L}} - 1$, and thus the price is

$$p^{(1)} = bL + R^D + k + c\sqrt{2kL} = p^{(2)} + c\Delta > p^{(2)}.$$

A direct calculation shows that

$$f_B^{(2)} > f_B^{(1)} \quad\Longleftrightarrow\quad \Delta\Big(L+\frac{R^D}{b+c}\Big) \;>\; \frac{p^{(2)}R^D}{b(b+c)},$$

where $p^{(2)} = bL + R^D + k + cL$. Since the right-hand side is $O(1/b)$ while the left-hand side tends to the positive constant ΔL, the inequality holds for all sufficiently large b. Therefore, for large b,

$$f_B^{DT} \;\le\; \frac{L+\frac{R^D}{b+c}}{p^{DT}}, \qquad p^{DT} = bL + (1+c)k + R^D = p^{SOE} + R^D.$$

and satisfies $\log(1+x) - \log(1+y) \le x - y$. Using

$$x = \frac{R^D}{(b+c)L}, \qquad y = \frac{R^D}{(b+c)L+k},$$

we obtain

$$\Phi(b) \le bkR^D \left[\frac{1}{(b+c)L} - \frac{1}{(b+c)L+k} \right] = bkR^D \frac{k}{\big((b+c)L\big)\big((b+c)L+k\big)}.$$

Since $b \le b+c$ and $(b+c)L+k \ge (b+c)L$,

$$\frac{b}{\big((b+c)L\big)\big((b+c)L+k\big)} \le \frac{b}{(b+c)^2L^2} \le \frac{1}{(b+c)L^2}.$$

Thus

$$\Phi(b) \le R^D k^2 \cdot \frac{1}{L^2} \cdot \frac{1}{b+c},$$

which proves the lemma. □

Remark 6. The maximum welfare gain of the traders from moving from the SOE regime to the distortionary–tax regime can be bounded by the regime when $\tau = 1$. This is because the welfare of the traders is decreasing in them being taxed. When $\tau = 1$, the gain for the rich traders is:

$$ck \left[\log\left(\frac{k}{p_{DT}}\right) - \log\left(\frac{L}{p_{SOE}}\right) \right].$$

Using the inequality $\log t \le t - 1$ for $t > 0$, we obtain the non–logarithmic bound:

$$ck \left[\log\left(\frac{k}{p_{DT}}\right) - \log\left(\frac{L}{p_{SOE}}\right) \right] \;\le\; ck \left(\frac{k}{L}\frac{p_{SOE}}{p_{DT}} - 1 \right) \le ck \left(\frac{k}{L} - 1 \right).$$

Deadweight loss. The deadweight loss from distortionary taxation equals the lost gains from luxury–luxury trade. Only luxuries traded for essentials remain efficiently allocated. In particular, each trader converts exactly

$$(1+\tau) \left[\frac{2k}{1+\tau} - L \right] = 2k - (1+\tau)L$$

units of their luxury endowment into essentials. Hence the deadweight loss is

$$\text{DWL} = \frac{c}{4} S(b) \;-\; c\big(2k - (1+\tau)L\big).$$

Since τ is bounded below and $S(b) \to \infty$, the deadweight loss grows without bound as $S(b)$

increases.

Putting it all together.

The net welfare effect of switching from SOE to the distortionary–tax regime is

$$W^{\mathrm{DT}} - W^{\mathrm{SOE}} = \underbrace{\Phi(b,c)}_{\text{gain to the poor}} + \underbrace{\psi(b,c)}_{\text{gain to the rich}} - \underbrace{\mathrm{DWL}}_{\text{lost surplus in luxury–luxury trade}} .$$

By Lemma 4.6,

$$\Phi(b,c) \;\leq\; \frac{R^{\mathrm{D}}k^2}{L^2} \cdot \frac{1}{b+c}.$$

From the bound on the traders' welfare gain,

$$\psi(b,c) \;<\; ck\left(\frac{k}{L} - 1\right),$$

which is finite and independent of b.

On the other hand, as $b \to \infty$ we assume $S(b) \to \infty$, and the deadweight loss

$$\mathrm{DWL} = \frac{c}{4}S(b) \;-\; c\big(2k - (1+\tau)L\big)$$

grows without bound. Hence

$$\lim_{b\to\infty} \left(W^{\mathrm{DT}} - W^{\mathrm{SOE}}\right) = -\infty.$$

Therefore there exists B such that for all $b \geq B$,

$$W^{\mathrm{DT}} - W^{\mathrm{SOE}} < 0,$$

even under the distortionary–tax redistribution regime.

A.4 Proof of Theorem 2

The utility functions are now:

$$U(.) = \frac{1}{\sigma}f^{\sigma} + \gamma_1 s_1 + \gamma_2 s_2 + l$$

$$U(.) = \frac{1}{\sigma}f^{\sigma} + \gamma_2 s_1 + \gamma_1 s_2 + l$$

We assume that $\sigma < 1$ and WLOG $\gamma_1 > \gamma_2 > 0$. As before, we assume the poor consume only essentials in an LF equilibrium (which is always true for large b and finite L).

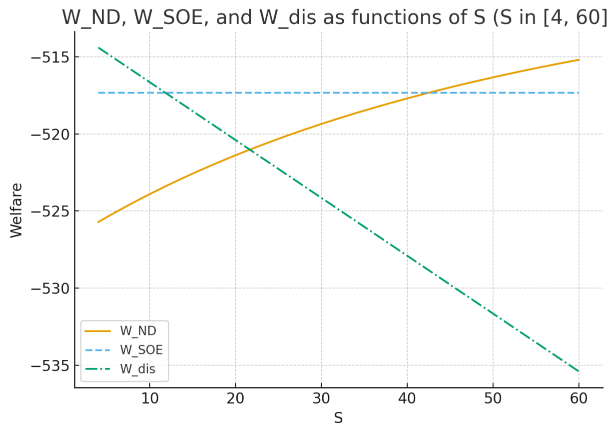

Figure 4: The y-axis measures welfare under SOE, welfare under distortionary taxes and welfare under non-distortionary taxes, and the x-axis measures the luxury endowment of the rich traders S. Utility functions for two types of agent: $k ln f + l + \gamma_1 s_1 + \gamma_2 s_2$ and $k ln f + l + \gamma_2 s_1 + \gamma_1 s_2$ where $k = 4.5, \gamma_1 = 1/2, \gamma_2 = 1/4$.. Endowment of leisure for everyone is $L = 1$. Relative number of landowners $a = 1$. Relative number of landless laborers $b = 30$. Relative number of rich traders $c = 1.5$.

Given the utility function, one can compute the demand for food grains given a price p and income level m.

$$f^* = \begin{cases} \frac{m}{p}, & \text{if } m - p^{\frac{\sigma}{\sigma-1}} < 0. \\ p^{\frac{1}{\sigma-1}}, & \text{if } m - p^{\frac{\sigma}{\sigma-1}} \geq 0. \end{cases}$$

To prove Theorem 2, we first prove that for any tax rate τ and corresponding revenue R, transfers are better than subsidies for utilitarian welfare.

Lemma 4.7. *For any tax rate τ and corresponding revenue R, if $\sigma < 0$, there exists $\hat{B} = \hat{B}(k, L, c, \sigma)$ such that for all $b > \hat{B}$, direct transfers yield higher total welfare than subsidies.*

Proof. We analyze the case where b is sufficiently large that all agents are at interior solutions. First notice that by assumption in the theorem,

$$\frac{S(b)}{b^{-\sigma}} \to \infty$$

This implies that the endowment of the rich $S(b)$ is of the order greater than $b^{-\sigma}$. Moreover, for large enough b, the (unconstrained) expenditure on foodgrains for the rich is of $p^{\frac{\sigma}{\sigma-1}}$. Since, p is of the same order as b, expenditure for the rich on essentials is of smaller order than their endowment which is of the order b^{σ}. Hence, traders are never income constrained.

Thus, for the landless poor (B) and the traders (C) and the landlords (A) is,

$$f_B = \frac{L}{p}, \qquad f_A = f_C = p^{\frac{1}{\sigma-1}}.$$

Let

$$\alpha \equiv \frac{1}{\sigma - 1} \in (-1, 0),$$

Direct transfers (DT). Under direct transfers, each poor agent receives an additional transfer R/b, so individual income is $m^{\mathrm{DT}} = L + R/b$. Aggregate poor income is $Q^{\mathrm{DT}} = bL + R$.

Asymptotic expansion for the DT price.

Under direct transfers (DT), the equilibrium price p^{DT} solves an equation of the form

$$(1+c)(p^{DT})^{\alpha} + \frac{Q}{p^{DT}} = 1, \tag{17}$$

where $Q = bL + R$. We are interested in the asymptotic behaviour of p^{DT} as $Q \to \infty$.

Given $\alpha < 0$, so that p^{α} becomes small when p is large. For large Q, the term Q/p forces p to be of order Q. We therefore set

$$p = Q + \delta, \qquad \delta = o(Q),$$

and expand both p^{α} and Q/p in powers of δ/Q. Matching terms in powers of Q^{α} yields

$$\delta = (1+c)\,Q^{\alpha+1} + (1+c)^2(\alpha+1)\,Q^{2\alpha+1} + o\big(Q^{2\alpha+1}\big),$$

so that the large-Q expansion of p is

$$p(Q) = Q + (1+c)\,Q^{\alpha+1} + (1+c)^2(\alpha+1)\,Q^{2\alpha+1} + o\big(Q^{2\alpha+1}\big), \qquad Q \to \infty,\ \alpha < 0. \tag{18}$$

Equivalently, factorising Q,

$$p(Q) = Q\left[1 + (1+c)\,Q^{\alpha} + (1+c)^2(\alpha+1)\,Q^{2\alpha} + o\big(Q^{2\alpha}\big)\right]. \tag{19}$$

$$\frac{1}{p(Q)} = \frac{1}{Q} - (1+c)\,Q^{\alpha-1} - \alpha(1+c)^2\,Q^{2\alpha-1} + o\big(Q^{2\alpha-1}\big).$$

The landless consumption under DT equals

$$f_B^{DT} = \frac{1}{b}\frac{Q}{p^{DT}} = \frac{1}{b}\Big[\,1 - (1+c)Q^{\alpha} - \alpha(1+c)^2 Q^{2\alpha} + o(Q^{2\alpha})\Big],$$

with $Q = bL + R$. Expanding for $b \to \infty$ (treating R as fixed) and collecting powers of b yields

$$f_B^{DT} = \frac{1}{b} - (1+c)L^{\alpha}\, b^{\alpha-1} - \alpha(1+c)^2 L^{2\alpha}\, b^{2\alpha-1} \tag{20}$$

$$- (1+c)\alpha L^{\alpha-1} R\, b^{\alpha-2} - 2\alpha^2(1+c)^2 L^{2\alpha-1} R\, b^{2\alpha-2} + o\big(b^{2\alpha-2}\big). \tag{21}$$

Note that the R-independent coefficients appear at orders b^{-1}, $b^{\alpha-1}$, and $b^{2\alpha-1}$; the first R-dependent contribution enters only at order $b^{\alpha-2}$. Since $\alpha \in (-1, 0)$, the powers satisfy $b^{-1} \gg b^{\alpha-1} \gg b^{2\alpha-1}$, so the expansion is well-ordered.

Subsidies (SB). Under subsidies the poor receive income $m^{\text{SB}} = L$ each. The equilibrium price p^{SB} satisfies

$$p^{\text{SB}} - \big((1+\tau_s)^{\beta} + c\big)\,(p^{\text{SB}})^{\beta} \;=\; bL, \qquad \beta = \frac{\sigma}{\sigma-1} = 1+\alpha,$$

with $\alpha \in (-1, 0)$. The subsidy revenue identity implies

$$R \;=\; p^{\text{SB}}\,\tau_s\Big(1 - \big((1+\tau_s)p^{\text{SB}}\big)^{\alpha}\Big).$$

Since $\alpha < 0$, $\big((1+\tau_s)p^{\text{SB}}\big)^{\alpha} = O\big((bL)^{\alpha}\big) \to 0$ as $b \to \infty$. Hence to leading order $\tau_s \sim R/p^{\text{SB}}$, and in particular

$$\tau_s \;=\; \frac{R}{bL} + o\big(b^{\alpha-2}\big),$$

where the error term is negligible for the expansion below (recall $b^{\alpha-2}$ decays faster than b^{-1} because $\alpha > -1$).

Expand $(1+\tau_s)^{\beta}$ for small τ_s:

$$(1+\tau_s)^{\beta} = 1 + \beta\tau_s + O(\tau_s^2) = 1 + \beta\frac{R}{bL} + o\big(b^{\alpha-2}\big).$$

Thus

$$D \equiv (1+\tau_s)^{\beta} + c = 1 + c + \beta\frac{R}{bL} + o\big(b^{\alpha-2}\big).$$

Writing $p^{\text{SB}} = bL + \delta$ with $\delta = o(bL)$ and solving iteratively (as in the DT case) gives

$$\begin{aligned}\frac{1}{p^{\text{SB}}} &= \frac{1}{bL} - D\,L^{\alpha-1}\,b^{\alpha-1} - \beta\,R\,L^{\alpha-2}\,b^{\alpha-2} + o\big(b^{\alpha-2}\big)\\ &= \frac{1}{bL} - (1+c)\,L^{\alpha-1}\,b^{\alpha-1} - (1+\alpha)\,R\,L^{\alpha-2}\,b^{\alpha-2} + o\big(b^{\alpha-2}\big),\end{aligned}$$

where in the second line we substituted $\beta = 1 + \alpha$ and used $D = 1 + c + O(b^{-1})$.

The consumption of the poor now equals:

$$f_B^{\mathrm{SB}} = \frac{L}{p^{\mathrm{SB}}} = \frac{1}{b} - (1+c)L^{\alpha}\, b^{\alpha-1} - (1+\alpha)R\, L^{\alpha-1}\, b^{\alpha-2} + o\big(b^{\alpha-2}\big). \tag{22}$$

Consumption and welfare comparison. Subtracting (22) from (20) gives

$$f_B^{\mathrm{DT}} - f_B^{\mathrm{SB}} = (1 - c\alpha)RL^{\alpha-1}b^{\alpha-2} + o\big(b^{\alpha-2}\big).$$

Since $\alpha = 1/(\sigma - 1) \in (-1, 0)$, we have $\alpha < 0$ and hence $(1 - c\alpha)RL^{\alpha-1} > 0$. Moreover, the $o(b^{\alpha-2})$ term is negligible relative to $b^{\alpha-2}$. Therefore there exists $\hat{B} = \hat{B}(k, L, c, \sigma)$ such that for all $b > \hat{B}$,

$$f_B^{\mathrm{DT}} - f_B^{\mathrm{SB}} > 0.$$

The total gain to the poor is of the order:

$$bu'(f)[f_B^{\mathrm{DT}} - f_B^{\mathrm{SB}}] = -b(1/b)^{\sigma-1}c\alpha RL^{\alpha-1}b^{\alpha-2}$$

the order of this object is $b^{\alpha-\sigma}$.

Welfare difference for the rich.

The rich consume

$$f(p) = p^{\alpha}, \qquad \alpha = \frac{1}{\sigma - 1} \in (-1, 0),$$

so

$$\frac{df}{dp} = \alpha\, p^{\alpha-1}.$$

A first–order expansion therefore gives

$$f(p_{DT}) - f(p_{SB}) \;\approx\; \alpha\, p^{\alpha-1}\, \Delta p, \qquad \Delta p \equiv p_{DT} - p_{SB},$$

where p may be evaluated at any intermediate point (we take $p \simeq p_{DT}$).

Under $u(f) = f^{\sigma}/\sigma$ we have

$$u'(f) = f^{\sigma-1} = (p^{\alpha})^{\sigma-1} = p.$$

Hence the welfare change per rich agent is

$$u'(f)\,\big(f(p_{DT}) - f(p_{SB})\big) \;\approx\; p \cdot \alpha\, p^{\alpha-1}\, \Delta p \;=\; \alpha\, p^{\alpha}\, \Delta p.$$

Multiplying by the mass c of rich agents yields the aggregate rich welfare difference

$$\Delta U_{\text{rich}} \;\approx\; c\,\alpha\,p^{\alpha}\,\Delta p.$$

Using the large-b expansions $p_{DT} \simeq bL + R$, $p_{SB} \simeq bL$, we have

$$\Delta p = p_{DT} - p_{SB} \simeq R, \qquad p^{\alpha} \simeq (bL)^{\alpha},$$

so that

$$\Delta U_{\text{rich}} \;\approx\; c\,\alpha\,R\,L^{\alpha}\,b^{\alpha}.$$

Thus the gain to the rich from DT relative to SB is of order b^{α}.

However since $\alpha - \sigma > \alpha$ for $\sigma < 0$, DT is better than SB for utilitarian welfare.

Distortionary taxes. Let $\phi = \sigma/(\sigma-1) = 1+\alpha \in (0,1)$. With a distortionary tax τ^D the DT price equation becomes

$$p - p^{\phi} - c\big((1+\tau^D)p\big)^{\phi} = bL + R^D \equiv Q.$$

Since $\phi - 1 = \alpha \in (-1,0)$ we have $p^{\phi-1} = p^{\alpha} \to 0$ as $p \to \infty$, so dividing by p yields $Q/p = 1 + o(1)$ and hence $p = Q(1+o(1))$. The subsidy/tax revenue satisfies

$$R^D = p\,\tau^D\Big(1 - ((1+\tau^D)p)^{\alpha}\Big),$$

so $\tau^D \sim R^D/p \sim R^D/Q$ and the same iterative expansion as in the non-distortionary case applies. If R^D is of the stated order and the tax affects the DT and SB equations symmetrically (so that the potentially larger $b^{2\alpha-1}$ terms cancel in the difference), then repeating the algebra above gives

$$f_B^{DT} - f_B^{SB} = (1 - c\alpha)\,R^D\,L^{\alpha-1}\,b^{\alpha-2} + o\big(b^{\alpha-2}\big),$$

and because $1 - c\alpha > 0$ (for $\alpha \in (-1,0)$ and $c \geq 0$) DT again dominates SB considering the welfare of the poor, for all sufficiently large b.

The order of the gain for the poor remains higher than the order of the rich as in the non-distortionary case (for the same reasons as discussed earlier). Hence, considering utilitarian welfare, DT dominates SB, for all sufficiently large b.

□

Now we compare SOE to CT with direct transfers. Just like the case with log-linear preferences, our argument proceeds in two steps. We first consider the non-distortionary case.

Non-Distortionary Taxes: If taxes are low, particularly $\tau \leq \left(\frac{\gamma_1}{\gamma_2} - 1\right)$, then they are non-distortionary. The maximum non-distortionary tax is $\tau = \left(\frac{\gamma_1}{\gamma_2} - 1\right)$. In this case the revenue (R^{ND}) is independent of b. To see this note that the equilibrium prices of both luxuries good must be equal and normalizing the price of universal good (l) to be 1, the prices must be γ_1. Hence $R^{ND} = \frac{\tau^{ND}}{1+\tau^{ND}}\gamma_1 cS = (\gamma_1 - \gamma_2)cS$. Since we assume S is a function of b, we get $R = R(b)$.

Lemma 4.8 (Non-distortionary taxation vs SOE)**.** *Fix $L > 0$, $c > 0$, and $\sigma < 0$. Consider an economy with b landless laborers and c rich traders, each endowed with L units of labor, and one landlord with a normalized foodgrain endowment 1. Let W_b^{SOE} denote aggregate welfare under the benchmark spheres-of-exchange regime, and let $W_b^{\mathrm{ND}}(R)$ denote aggregate welfare under a non-distortionary commodity tax regime that raises revenue $R = R(b) \geq 0$, rebated lump-sum to the landless poor.*

Suppose the (non-distortionary) revenue satisfies

$$\frac{R(b)}{b} \to 0 \quad \text{as } b \to \infty.$$

Then there exists $\hat{B} = \hat{B}(L, c, \sigma)$ such that for all $b > \hat{B}$,

$$W_b^{\mathrm{SOE}} > W_b^{\mathrm{ND}}(R(b)).$$

Proof. Given $u(f) = f^\sigma/\sigma$ with $\sigma < 0$ and define

$$\alpha \equiv \frac{1}{\sigma - 1} \in (-1, 0).$$

Step 1: Price equations and general asymptotics. In any regime where all agents are at interior solutions, aggregate consumption of the essential satisfies

$$Q\,\frac{1}{p} + A\,p^\alpha = 1,$$

where Q is the total labor income of agents who buy essentials and $A > 0$ is the coefficient on the landlord/trader demand term for foodgrain.

A standard asymptotic argument (see also Lemma 4.7) shows that for $Q \to \infty$ the unique solution $p(Q)$ with $p(Q) \sim Q$ admits the expansion

$$p(Q) = Q + AQ^{1+\alpha} + O\big(Q^{1+2\alpha}\big), \tag{23}$$

and hence

$$\frac{1}{p(Q)} = \frac{1}{Q} - AQ^{\alpha-1} - A^2\alpha Q^{2\alpha-1} + O\big(Q^{3\alpha-1}\big). \tag{24}$$

Step 2: SOE vs non-distortionary taxation (prices and consumption). Under SOE, the b landless laborers and c rich traders each have income L, so total labor income is

$$Q^{\text{SOE}} = (b+c)L.$$

The landlord is the only agent with interior demand. Thus, $A^{\text{SOE}} = 1$ in (23), and

$$p^{\text{SOE}} = Q^{\text{SOE}} + (Q^{\text{SOE}})^{1+\alpha} + O\big((Q^{\text{SOE}})^{1+2\alpha}\big).$$

The landless consume

$$f_B^{\text{SOE}} = \frac{L}{p^{\text{SOE}}}.$$

and hence

$$\begin{aligned} f_B^{\text{SOE}} = \frac{L}{p^{\text{SOE}}} &= \frac{L}{Q^{\text{SOE}}} - L(Q^{\text{SOE}})^{\alpha-1} + O\big((Q^{\text{SOE}})^{2\alpha-1}\big). \\ &= \frac{1}{b+c} - L^{\alpha}(b+c)^{\alpha-1} + O\big(b^{2\alpha-1}\big). \end{aligned}$$

Under non-distortionary taxation, each landless laborer receives a transfer $R(b)/b$, so their income is $L + R(b)/b$.

$$Q^{\text{ND}} = bL + R(b) = bL\Big(1+\varepsilon_b\Big), \qquad \varepsilon_b \equiv \frac{R(b)}{bL} \to 0 \quad (b \to \infty).$$

In this regime, both the landlord and the c traders have interior CES demand for foodgrain, so the coefficient on p^{α} is $A^{\text{ND}} = 1 + c$. Using (24) with $Q = Q^{\text{ND}}$ and $A = 1 + c$,

$$\frac{1}{p^{\text{ND}}} = \frac{1}{Q^{\text{ND}}} - (1+c)(Q^{\text{ND}})^{\alpha-1} + O\big((Q^{\text{ND}})^{2\alpha-1}\big).$$

The landless consumption is

$$f_B^{\text{ND}} = \frac{L + R(b)/b}{p^{\text{ND}}} = \frac{Q^{\text{ND}}}{b}\frac{1}{p^{\text{ND}}}.$$

Plugging in $\frac{1}{p_{ND}}$, we get:

$$f_B^{\text{ND}} = \frac{1}{b} - (1+c)(Q^{\text{ND}})^{\alpha}/b + O\big((Q^{\text{ND}})^{2\alpha}/b\big). \tag{25}$$

Using $Q^{\mathrm{ND}} = bL(1+\varepsilon_b)$ and $(Q^{\mathrm{ND}})^\alpha = (bL)^\alpha\big(1+\alpha\varepsilon_b + o(\varepsilon_b)\big)$ with $\varepsilon_b \to 0$, (25) gives:

$$f_B^{\mathrm{ND}} = \frac{1}{b} - (1+c)L^\alpha b^{\alpha-1} + O\big(b^{\alpha-2}\big) + O\big(b^{2\alpha-1}\big).$$

Subtracting,

$$f_B^{\mathrm{SOE}} - f_B^{\mathrm{ND}} = cL^\alpha b^{\alpha-1} + O\big(b^{\alpha-2}\big) + O\big(b^{2\alpha-1}\big). \tag{26}$$

Since $c > 0$, $L > 0$, and $\alpha \in (-1, 0)$, we have $cL^\alpha > 0$ and $b^{\alpha-1} > 0$, so the leading term in (26) is strictly positive. Furthermore, because $\alpha \in (-1, 0)$ we have $\alpha - 1 > 2\alpha - 1$ and $\alpha - 1 > \alpha - 2$, so $b^{\alpha-1}$ dominates both $b^{2\alpha-1}$ and $b^{\alpha-2}$ as $b \to \infty$. Thus there exists B_1 such that for all $b > B_1$,

$$f_B^{\mathrm{SOE}} - f_B^{\mathrm{ND}} > 0.$$

Step 3: Welfare comparison and dominance of the landless. For each landless worker,

$$u'(f) = f^{\sigma-1} > 0,$$

and for large b, f_B^{SOE} and f_B^{ND} are both of order $1/b$. A first-order Taylor expansion around f_B^{ND} yields

$$u\big(f_B^{\mathrm{SOE}}\big) - u\big(f_B^{\mathrm{ND}}\big) = u'\big(f_B^{\mathrm{ND}}\big)\big(f_B^{\mathrm{SOE}} - f_B^{\mathrm{ND}}\big) + o\big(f_B^{\mathrm{SOE}} - f_B^{\mathrm{ND}}\big).$$

Since $f_B^{\mathrm{ND}} \sim 1/b$, we have $u'\big(f_B^{\mathrm{ND}}\big) \sim (1/b)^{\sigma-1} = b^{1-\sigma}$, and using (26) we obtain, for some constant $K > 0$,

$$u\big(f_B^{\mathrm{SOE}}\big) - u\big(f_B^{\mathrm{ND}}\big) = K\, b^{1-\sigma} \cdot b^{\alpha-1} + o\big(b^{\alpha-\sigma}\big) = K\, b^{\alpha-\sigma} + o\big(b^{\alpha-\sigma}\big),$$

with $K > 0$.

Aggregating over b landless workers, their total welfare gain is

$$\Delta W_B(b) \equiv b\Big[u\big(f_B^{\mathrm{SOE}}\big) - u\big(f_B^{\mathrm{ND}}\big)\Big] = K\, b^{1+\alpha-\sigma} + o\big(b^{1+\alpha-\sigma}\big).$$

Since $\alpha > -1$ and $\sigma < 0$, we have $1 + \alpha - \sigma > 0$, so $\Delta W_B(b) \to +\infty$.

We now check that the contributions from the c traders and the single landlord are asymptotically negligible relative to $\Delta W_B(b)$.

The order of f_C^{ND} is at max $p^\alpha = O(b^\alpha)$. f_C^{SOE} and f_C^{ND} are of order at most $\max\{(bL)^\alpha\}$, and similarly for f_A^{SOE} and f_A^{ND}. Therefore

$$\left|u(f_C^{\mathrm{SOE}}) - u(f_C^{\mathrm{ND}})\right| \leq u(f_C^{ND}), \qquad \left|u(f_A^{\mathrm{SOE}}) - u(f_A^{\mathrm{ND}})\right| \leq u(f_A^{ND})$$

both of which are of the order $[b^{\alpha}]^{\sigma}$ hence the order of the welfare difference is at max:

$$\Delta W_{C,A}(b) = O\big(b^{\alpha\sigma}\big).$$

Since $\alpha \in (-1, 0)$ and $\sigma < 0$, we have

$$1 + \alpha - \sigma > \alpha\sigma,$$

because

$$(1 + \alpha - \sigma) - \alpha\sigma = (1 + \alpha)(1 - \sigma) > 0.$$

Hence $b^{1+\alpha-\sigma}$ dominates $b^{\alpha\sigma}$ as $b \to \infty$, so $\Delta W_{C,A}(b)$ is negligible relative to $\Delta W_B(b)$.

Step 4: Conclusion. Combining the pieces,

$$W_b^{\mathrm{SOE}} - W_b^{\mathrm{ND}}(R(b)) = \Delta W_B(b) + \Delta W_{C,A}(b) = K\, b^{1+\alpha-\sigma} + o\big(b^{1+\alpha-\sigma}\big),$$

with $K > 0$. Therefore there exists $\hat{B}$ such that for all $b > \hat{B}$,

$$W_b^{\mathrm{SOE}} > W_b^{\mathrm{ND}}(R(b)).$$

□

Distortionary Taxes: $\tau > \frac{\gamma_1}{\gamma_2} - 1$.

Lemma 4.9 (Upper Bound on Welfare Gains from Distortionary Taxation)**.** *Fix* $L > 0$*,* $c > 0$*, and* $\sigma < 0$*, and let*

$$\alpha := \frac{1}{\sigma - 1} \in (-1, 0).$$

Consider an economy with b *landless laborers and* c *rich traders, each endowed with* L *units of labor, and one landlord with a normalized foodgrain endowment* 1*.*

Let W_b^{SOE} *denote aggregate welfare of the landless laborers and landowners under the benchmark spheres-of-exchange regime, and let* $W_b^{\mathrm{D}}(R)$ *denote the aggregate welfare of the landless laborers and landowners under* any *distortionary commodity tax regime that raises revenue* $R = R(b) \geq 0$*, rebated lump-sum.*

Suppose $\lim_{b\to\infty} \dfrac{R(b)}{b} = 0$. *Then there exists* $\hat{B}, C(L,c,\sigma) > 0$ *such that, for all* $b > \hat{B}$,

$$W_b^{\mathrm{D}}(R(b)) - W_b^{\mathrm{SOE}} \;\leq\; C(L,c,\sigma)\, b^{\alpha-\sigma} R(b) \;+\; o\big(b^{\alpha-\sigma} R(b)\big).$$

In particular, the maximum possible welfare gain *from any distortionary tax regime that raises revenue* $R(b)$ *is at most of order*

$$W_b^{\mathrm{D}}(R(b)) - W_b^{\mathrm{SOE}} = O\big(b^{\alpha-\sigma} R(b)\big) = O\Big(b^{\frac{1}{\sigma-1}-\sigma} R(b)\Big).$$

Proof. Step 1: An upper–bound regime. Fix b, c, L, σ and let $R = R(b) \geq 0$ be any given revenue available for redistribution. Under an arbitrary distortionary tax regime denote by $f_A^{\mathrm{D}}, f_B^{\mathrm{D}}, f_C^{\mathrm{D}}$ the food consumptions of the landlord, the b landless laborers, and the c rich traders, respectively. By standard monotonicity of demand in income (food is a normal good) and since rich traders have weakly higher effective income than the landless, we have

$$f_B^{\mathrm{D}} \leq f_C^{\mathrm{D}}.$$

To obtain an upper bound on the welfare achievable for the landless under any distortionary scheme that raises revenue R, consider the following *idealized* allocation (the *upper–bound* regime). Give each non-landlord an equal lump-sum transfer of $R/(b+c)$ (so each landless worker and each trader has income $L + R/(b+c)$), and let the landlord demand according to the CES first-order condition (given landowners are endowed with essential goods they can't be constrained in their demand). Let p^{UB} denote the market price that clears in this allocation and write f^{UB} for the common food consumption of each non-landlord:

$$f^{\mathrm{UB}} \;=\; \frac{L + \dfrac{R}{b+c}}{p^{\mathrm{UB}}}.$$

By construction this allocation is feasible (it uses exactly the given revenue R and clears the market) and it equalizes consumption across all non-landlords. Consequently, for any distortionary regime that raises the same revenue R we must have

$$W_b^{\mathrm{D}}(R) \leq \overline{W}_b^{\mathrm{D}}(R),$$

where $\overline{W}_b^{\mathrm{D}}(R)$ is the aggregate welfare for the poor in the upper–bound regime. It therefore suffices to bound $\overline{W}_b^{\mathrm{D}}(R(b)) - W_b^{\mathrm{SOE}}$.

Step 2: Equilibrium conditions under SOE and in the upper-bound regime. Let $X_b :=$

$(b+c)L$ denote aggregate labor income when all non-landlords have income L. Under SOE, non-landlords cannot use luxury wealth to purchase essentials, so each of the $b+c$ non-landlords has income L and the landlord has CES demand for foodgrain. The equilibrium price p^{SOE} solves

$$X_b \;=\; p^{\text{SOE}} - \big(p^{\text{SOE}}\big)^{\frac{\sigma}{\sigma-1}}. \tag{27}$$

In the upper-bound distortionary regime, all $b+c$ non-landlords receive an extra transfer $R(b)/(b+c)$, so each has income $L + R(b)/(b+c)$. Aggregate labor income of non-landlords is $X_b + R(b)$, and the landlord's food demand is again interior CES. The equilibrium price p^{UB} solves

$$X_b + R(b) \;=\; p^{\text{UB}} - \big(p^{\text{UB}}\big)^{\frac{\sigma}{\sigma-1}}. \tag{28}$$

Step 3: Asymptotic price expansions. Consider the implicit equation

$$X = p - p^{1+\alpha}, \qquad \alpha = \frac{1}{\sigma-1} \in (-1,0),$$

for large X. A standard implicit-function expansion yields

$$p(X) = X + X^{1+\alpha} + O\big(X^{1+2\alpha}\big), \qquad X \to \infty.$$

Thus, as $b \to \infty$,

$$p^{\text{SOE}} = X_b + X_b^{1+\alpha} + O\big(X_b^{1+2\alpha}\big), \tag{29}$$

$$p^{\text{UB}} = X_b + R(b) + \big(X_b + R(b)\big)^{1+\alpha} + O\big((X_b + R(b))^{1+2\alpha}\big). \tag{30}$$

Since $R(b) = o(X_b)$ by assumption, we have $X_b + R(b) \sim X_b$ and $(X_b + R(b))^{1+\alpha} \sim X_b^{1+\alpha}$.

Step 4: Non-landlord consumption and its difference. Under SOE, non-landlords consume

$$f^{\text{SOE}} = \frac{L}{p^{\text{SOE}}},$$

while in the upper-bound regime, each non-landlord consumes

$$f^{\text{UB}} = \frac{L + R(b)/(b+c)}{p^{\text{UB}}}.$$

Using (29) with $p^{\text{SOE}} = X_b(1 + X_b^{\alpha} + o(X_b^{\alpha}))$, we get

$$\frac{1}{p^{\text{SOE}}} = \frac{1}{X_b}\Big(1 - X_b^{\alpha} + O\big(X_b^{2\alpha}\big)\Big),$$

so

$$f^{\mathrm{SOE}} = \frac{L}{X_b}\Big(1 - X_b^{\alpha} + O\big(X_b^{2\alpha}\big)\Big). \tag{31}$$

For the upper-bound regime, write $X_b + R(b) = X_b(1+\varepsilon_b)$ with $\varepsilon_b = R(b)/X_b \to 0$. Then from (30),

$$p^{\mathrm{UB}} = X_b(1+\varepsilon_b)\Big(1 + (X_b + R(b))^{\alpha} + O\big((X_b + R(b))^{2\alpha}\big)\Big).$$

Thus

$$\frac{1}{p^{\mathrm{UB}}} = \frac{1}{X_b(1+\varepsilon_b)}\Big(1 - (X_b + R(b))^{\alpha} + O\big((X_b + R(b))^{2\alpha}\big)\Big).$$

Since

$$L + \frac{R(b)}{b+c} = \frac{X_b + R(b)}{b+c} = \frac{X_b(1+\varepsilon_b)}{b+c},$$

we get

$$f^{\mathrm{UB}} = \frac{X_b(1+\varepsilon_b)}{b+c}\cdot\frac{1}{p^{\mathrm{UB}}} = \frac{1}{b+c}\Big(1 - (X_b + R(b))^{\alpha} + O\big((X_b + R(b))^{2\alpha}\big)\Big). \tag{32}$$

Subtracting (31) (rewritten in the same $1/(b+c)$ form) from (32) yields

$$\begin{aligned} f^{\mathrm{UB}} - f^{\mathrm{SOE}} &= \frac{1}{b+c}\Big[X_b^{\alpha} - (X_b + R(b))^{\alpha}\Big] + O\big(X_b^{2\alpha-1}\big) \\ &= -\frac{1}{b+c}\alpha X_b^{\alpha-1}R(b) + o\big(X_b^{\alpha-1}R(b)\big), \end{aligned} \tag{33}$$

where we used the expansion

$$(X_b + R(b))^{\alpha} = X_b^{\alpha} + \alpha X_b^{\alpha-1}R(b) + o\big(X_b^{\alpha-1}R(b)\big) \quad \text{as } b \to \infty.$$

Since $\alpha < 0$, the leading coefficient in (33) is positive, confirming that $f^{\mathrm{UB}} > f^{\mathrm{SOE}}$ for large b.

Step 5: The landless laborers welfare gain. With CRRA utility $u(f) = f^{\sigma}/\sigma$ and $\sigma < 0$, we have $u'(f) = f^{\sigma-1} > 0$. For f^{UB} and f^{SOE} both of order $1/(b+c)$, a first-order Taylor expansion gives

$$u\big(f^{\mathrm{UB}}\big) - u\big(f^{\mathrm{SOE}}\big) = u'\big(f^{\mathrm{SOE}}\big)\big(f^{\mathrm{UB}} - f^{\mathrm{SOE}}\big) + o\big(f^{\mathrm{UB}} - f^{\mathrm{SOE}}\big).$$

Since $f^{\mathrm{SOE}} \sim 1/(b+c)$, we have

$$u'\big(f^{\mathrm{SOE}}\big) \sim \left(\frac{1}{b+c}\right)^{\sigma-1} = (b+c)^{1-\sigma}.$$

Combining with (33), we obtain

$$u\big(f^{\mathrm{UB}}\big) - u\big(f^{\mathrm{SOE}}\big) = K_1(L,\sigma)\,(b+c)^{1-\sigma} X_b^{\alpha-1} R(b) + o\big((b+c)^{1-\sigma} X_b^{\alpha-1} R(b)\big),$$

for some positive constant $K_1(L,\sigma) = -\alpha L^{\alpha-1}$ (up to harmless positive factors).

Summing over the b landless laborers, and ignoring the finite mass of c traders, the total welfare gain is

$$\begin{aligned}\Delta W_{\mathrm{B}}(b) &:= b\Big[u\big(f^{\mathrm{UB}}\big) - u\big(f^{\mathrm{SOE}}\big)\Big] \\ &= K_2(L,\sigma)\,(b)^{2-\sigma} X_b^{\alpha-1} R(b) + o\big((b)^{2-\sigma} X_b^{\alpha-1} R(b)\big).\end{aligned}$$

Since $X_b = (b+c)L \approx bL$, we have

$$(b)^{2-\sigma} X_b^{\alpha-1} = (b)^{2-\sigma}(b)^{\alpha-1}L^{\alpha-1} = L^{\alpha-1}(b)^{\alpha-\sigma},$$

so

$$\Delta W_{\mathrm{B}}(b) = K_3(L,\sigma)\,(b)^{\alpha-\sigma} R(b) + o\big(b^{\alpha-\sigma} R(b)\big), \tag{34}$$

for some $K_3(L,\sigma) = -\alpha L^{\alpha-1} > 0$.

Step 6: Landlord welfare term The landlord's food consumption under SOE and under the upper-bound regime satisfies

$$f_A^{\mathrm{SOE}} = \big(p^{\mathrm{SOE}}\big)^{\alpha}, \qquad f_A^{\mathrm{UB}} = \big(p^{\mathrm{UB}}\big)^{\alpha}.$$

Using (29)–(30) and the same binomial expansion as above, we obtain

$$f_A^{\mathrm{UB}} - f_A^{\mathrm{SOE}} = O\big(X_b^{\alpha-1} R(b)\big),$$

and hence

$$u\big(f_A^{\mathrm{UB}}\big) - u\big(f_A^{\mathrm{SOE}}\big) = O\big(X_b^{\alpha\sigma-1} R(b)\big).$$

Since $\alpha \in (-1,0)$ and $\sigma < 0$, one can verify that

$$\alpha - \sigma > \alpha\sigma - 1,$$

so $b^{\alpha-\sigma}R(b)$ dominates $b^{\alpha\sigma-1}R(b)$ as $b \to \infty$. Thus the landlord's welfare term is of strictly lower order than (34).

Step 7: Final bound. Combining the landless laborers and landlord contributions, there exists $C(L,c,\sigma) > 0$ such that

$$\overline{W}_b^{\mathrm{D}}(R(b)) - W_b^{\mathrm{SOE}} \;\le\; C(L,c,\sigma)\, b^{\alpha-\sigma} R(b) \;+\; o\big(b^{\alpha-\sigma} R(b)\big).$$

Finally, since $W_b^{\mathrm{D}}(R(b)) \le \overline{W}_b^{\mathrm{D}}(R(b))$ for all b, the same upper bound holds for any distortionary tax regime, which completes the proof.

□

We now compute the maximum value of R which the planner could generate. Basically, we compute the order of the revenue-maximizing tax and revenue for large p. We show that both the tax and the revenue is of the order $b^{-\sigma}$.

Lemma 4.10. *Let*

$$T(\tau) = \tau\Big(p\,[p(1+\tau)]^{\frac{1}{\sigma-1}} - L\Big) = \tau\Big(A(1+\tau)^{\alpha} - L\Big), \qquad A := p^{\frac{\sigma}{\sigma-1}}, \quad \alpha := \frac{1}{\sigma-1},$$

with constants $L > 0$ and $\sigma < 0$ (hence $\alpha \in (-1,0)$). Assume $p = \Theta(b)$ as $b \to \infty$. Then the unique revenue-maximising tax $\tau^\star$ satisfies

$$\tau^\star = \Theta\big(b^{-\sigma}\big), \qquad T(\tau^\star) = \Theta\big(b^{-\sigma}\big).$$

Proof. (i) Uniqueness and interiority. We have

$$T'(\tau) = A(1+\tau)^{\alpha} - L + A\alpha\,\tau(1+\tau)^{\alpha-1} = A(1+\tau)^{\alpha-1}\big[1+(1+\alpha)\tau\big] - L.$$

Since $\alpha < 0$, we have $\lim_{\tau\to\infty} T'(\tau) = -L < 0$ and $T'(0) = A - L$. Because $p \to \infty$ and $\sigma < 0$, we have $A = p^{\sigma/(\sigma-1)} \to \infty$, so for all sufficiently large p, $T'(0) > 0$. By continuity there is a unique $\tau^\star > 0$ with $T'(\tau^\star) = 0$, and

$$T''(\tau) = A\alpha(1+\tau)^{\alpha-2}\big[2+(1+\alpha)\tau\big] < 0$$

for all $\tau \ge 0$, which implies $\tau^\star$ is the unique global maximiser.

(ii) Asymptotics of $\tau^\star$. The first-order condition can be written as

$$A(1+\tau^\star)^{\alpha-1}\big[1+(1+\alpha)\tau^\star\big] = L. \tag{35}$$

Since $\sigma < 0$ and $A \to \infty$, the solution of (35) must satisfy $\tau^\star \to \infty$. Hence

$$1+(1+\alpha)\tau^\star \sim (1+\alpha)\tau^\star, \qquad (1+\tau^\star)^{\alpha-1} \sim (\tau^\star)^{\alpha-1},$$

and (35) implies

$$A(1+\alpha)(\tau^\star)^\alpha \;\sim\; L \quad\Longrightarrow\quad \tau^\star \;\sim\; \left(\tfrac{L}{A(1+\alpha)}\right)^{1/\alpha}. \tag{36}$$

Since $A = p^{\sigma/(\sigma-1)}$ and $1/\alpha = \sigma - 1$,

$$\tau^\star \sim \left(\tfrac{L}{1+\alpha}\right)^{\sigma-1} A^{-(\sigma-1)} = \left(\tfrac{L}{1+\alpha}\right)^{\sigma-1} p^{-\sigma},$$

so $\tau^\star = \Theta(p^{-\sigma})$. With $p = \Theta(b)$, this gives $\tau^\star = \Theta(b^{-\sigma})$.

(iii) Asymptotics of maximal revenue. Using the FOC (35) to eliminate L we obtain

$$T(\tau^\star) = \tau^\star\big[A(1+\tau^\star)^\alpha - L\big] = -\alpha\, A\,(\tau^\star)^2(1+\tau^\star)^{\alpha-1} \sim -\alpha\, A\,(\tau^\star)^{1+\alpha}.$$

Substituting $\tau^\star \sim \big(L/(A(1+\alpha))\big)^{1/\alpha}$ yields

$$T(\tau^\star) \sim -\alpha\, L^{\frac{1+\alpha}{\alpha}}(1+\alpha)^{-\frac{1+\alpha}{\alpha}}\, A^{-\frac{1}{\alpha}}.$$

Noting $A^{-\frac{1}{\alpha}} = p^{-\sigma}$ and defining

$$C(\sigma, L) := -\alpha\, L^{\frac{1+\alpha}{\alpha}}(1+\alpha)^{-\frac{1+\alpha}{\alpha}} > 0,$$

we conclude

$$T(\tau^\star) \sim C(\sigma, L)\, p^{-\sigma} = \Theta(p^{-\sigma}).$$

With $p = \Theta(b)$ this gives $T(\tau^\star) = \Theta(b^{-\sigma})$, as claimed. □

Combining Lemma 4.9 and Lemma 4.10, and recalling that $\alpha = \frac{1}{\sigma-1}$, we obtain the following corollary.

Since Lemma 4.9 implies

$$W_b^{\mathrm{D}}(R(b)) - W_b^{\mathrm{SOE}} \;\le\; C(L,c,\sigma)\, b^{\alpha-\sigma} R(b) \;+\; o\big(b^{\alpha-\sigma} R(b)\big),$$

and Lemma 4.10 yields $R(b) = T(\tau^\star) = \Theta(b^{-\sigma})$, we have, for some constant $\widetilde{C}(L,c,\sigma) > 0$,

$$\sup_{\text{distortionary tax regimes}} \big(W_b^{\mathrm{D}}(R) - W_b^{\mathrm{SOE}}\big) \;\le\; \widetilde{C}(L,c,\sigma)\, b^{\alpha-2\sigma} \;+\; o\big(b^{\alpha-2\sigma}\big).$$

Equivalently,

$$\sup_{\text{distortionary tax regimes}} \big(W_b^{\mathrm{D}}(R) - W_b^{\mathrm{SOE}}\big) = O\Big(b^{\frac{1}{\sigma-1}-2\sigma}\Big).$$

In words, even the best-designed distortionary commodity tax can improve welfare relative to the spheres-of-exchange benchmark by at most a term of order $b^{\frac{1}{\sigma-1}-2\sigma}$.

Lemma 4.11. *Let $\sigma < 0$ and suppose the luxury endowment function $S : \mathbb{N} \to \mathbb{R}_+$ satisfies*

$$\lim_{b\to\infty} \frac{S(b)}{b^{-\sigma}} = \infty.$$

then the deadweight loss from distortionary taxation dominates any gain to the rich traders. On the other hand if:

$$\lim_{b\to\infty} \frac{S(b)}{b^{-\sigma}} = 0.$$

Then for all sufficiently large b the planner can appropriate all luxuries with a commodity tax, and there is no deadweight loss relative to the SOE *allocation.*

Proof. Step 1: Welfare gain to the traders.

In any distortionary taxation regime the price p is bounded below by the SOE price p_{SOE}. Hence the welfare gain to the traders can be bounded above by comparing unconstrained interior consumption at the SOE price to the SOE consumption at which they are forced to consume as much as the poor (the constrained corner allocation):

$$f_A(p_{\text{SOE}}) = p_{\text{SOE}}^{\frac{1}{\sigma-1}}, \qquad u(f) = \frac{f^\sigma}{\sigma}.$$

Thus

$$u(f_A(p_{\text{SOE}})) - u\left(\frac{L}{p_{\text{SOE}}}\right) = \frac{1}{\sigma}\left[p_{\text{SOE}}^{\frac{\sigma}{\sigma-1}} - L^\sigma p_{\text{SOE}}^{-\sigma}\right].$$

Since $\sigma < 0$, both exponents $\frac{\sigma}{\sigma-1}$ and $-\sigma$ are positive, and one checks that $-\sigma \geq \frac{\sigma}{\sigma-1}$. Hence the dominant term in the bracket for large p_{SOE} is $L^\sigma p_{\text{SOE}}^{-\sigma}$, so

$$u(f_A(p_{\text{SOE}})) - u\left(\frac{L}{p_{\text{SOE}}}\right) = \Theta\big(p_{\text{SOE}}^{-\sigma}\big).$$

Since $p_{\text{SOE}} \sim bL$, this is of order

$$(bL)^{-\sigma} = \Theta\big(b^{-\sigma}\big).$$

Because the deadweight loss from restricting the luxury sphere is of order $S(b)$, whenever $S(b) \gg b^{-\sigma}$ the rich agents' potential welfare gain is negligible relative to the DWL. Under the growth conditions in our theorem ($S(b)/b^{-\sigma} \to \infty$), the DWL term strictly dominates the rich agents' possible gains.

Step 2: No deadweight loss when luxury endowment grows slowly

By Lemma 4.10, in an economy with b landless traders and price $p = \Theta(b)$, the revenue-maximising commodity tax $\tau^\star(b)$ yields aggregate revenue

$$R_b^{\mathrm{D}}(\tau^\star(b)) \;=\; T(\tau^\star(b)) \;=\; \Theta\big(b^{-\sigma}\big).$$

If the following is true:

$$\frac{S(b)}{b^{-\sigma}} \;\longrightarrow\; 0$$

then for any $\varepsilon > 0$ there exists $\widehat{B}$ such that for all $b \geq \widehat{B}$,

$$S(b) \;\leq\; \varepsilon\, b^{-\sigma}.$$

Choosing ε small enough, we can ensure that for all sufficiently large b,

$$S(b) \;\leq\; R_b^{\mathrm{D}}(\tau^\star(b)).$$

Hence, for each $b \geq \widehat{B}$ there exists a (possibly smaller) tax rate $\tilde{\tau}(b) \in (0, \tau^\star(b)]$ such that the associated revenue $R(b) = T(\tilde{\tau}(b))$ exactly equals the total value of luxuries the planner wishes to appropriate.

Since preferences are linear in the luxury goods and the planner only uses the commodity tax to transfer their value (without distorting the subsistence-good allocation relative to the SOE benchmark), this confiscation of luxuries is non-distortionary. The planner can replicate the SOE allocation in the subsistence and labour–leisure dimensions and simply reassign the luxuries via lump-sum transfers.

Therefore, whenever $S(b)/b^{-\sigma} \to 0$, the planner can appropriate all luxuries with a commodity tax for all sufficiently large b, and there is no deadweight loss relative to SOE. □

Combining Lemmas 4.9, 4.10, and 4.11, we obtain the following asymptotic comparison. Lemma 4.9 and Lemma 4.10 imply that, even under an optimally designed distortionary commodity tax, the maximal welfare gain relative to SOE is at most of order $b^{\frac{1}{\sigma-1}-2\sigma}$. Lemma 4.11 shows that when $S(b)$ is not too small, the deadweight loss outweigh any gain for the rich and their endowment can't be appropriated.

Thus, if the luxury endowment satisfies

$$\frac{S(b)}{b^{\max\left(\frac{1}{\sigma-1}-2\sigma,\,-\sigma\right)}} \;\longrightarrow\; \infty,$$

then for large b the welfare losses from restricting the luxury sphere (of order $S(b)$) dominate any potential welfare gains from distortionary taxation (of order at most $b^{\frac{1}{\sigma-1}-2\sigma}$ or $b^{-\sigma}$), whichever is higher. In this sense, the SOE regime is asymptotically strictly welfare-superior.

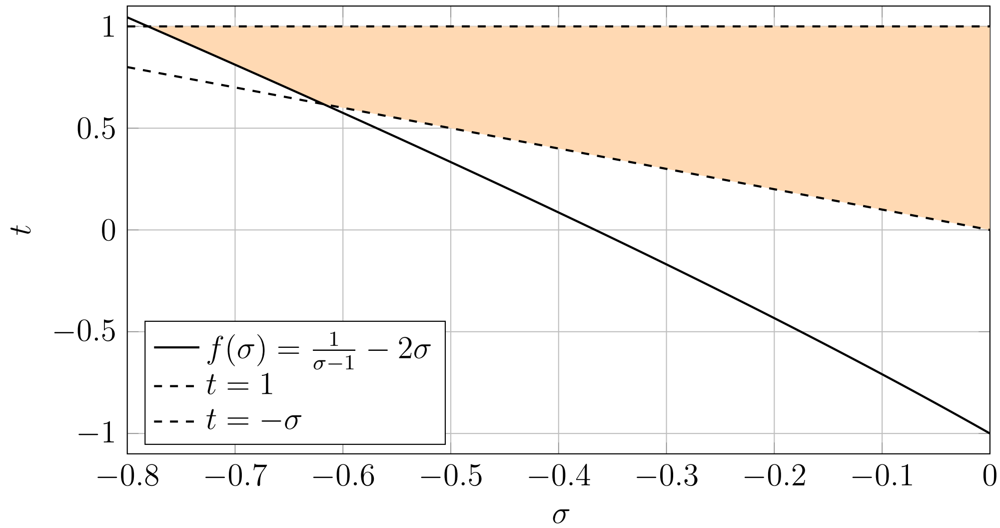

Figure 5: Suppose the economy with b landless agents has trader endowment (b^t, b^t). Preferences are: $U_1 = \frac{f^\sigma}{\sigma} + \gamma_1 s_1 + \gamma_2 s_2$ and $U_2 = \frac{f^\sigma}{\sigma} + \gamma_2 s_1 + \gamma_1 s_2$ with equal probability. The x-axis measures the substitution parameter σ and the y-axis measures t. The shaded region represents values of t such that for large enough b, SOE policies outperform the optimal tax. These hold for any value of $c, L > 0$.

Feasibility of the two growth conditions. We show that the two growth requirements

$$S(b) \to \infty, \qquad \frac{S(b)}{b} \to 0,$$

and

$$\frac{S(b)}{b^{\max\{\alpha-2\sigma,\, -\sigma\}}} \to \infty, \qquad \text{where } \alpha = \frac{1}{\sigma - 1},$$

can be satisfied simultaneously if $\sigma \in (-0.78, 0)$.

Indeed, for the existence of such an $S(b)$ it is necessary that the exponent $\max\{\alpha - 2\sigma,\, -\sigma\}$ be strictly less than 1. The inequality $-\sigma < 1$ is equivalent to $\sigma > -1$. The inequality $\alpha - 2\sigma < 1$ yields $-0.78 < \sigma < 0$. Hence the two constraints are jointly satisfiable exactly when $\sigma \in (-0.78, 0)$. As a concrete example, fix any $\sigma \in (-0.78, 0)$ and choose an exponent r with

$$\max\{\alpha - 2\sigma,\, -\sigma\} \;<\; r \;<\; 1.$$

Then $S(b) = b^r$ satisfies $S(b) \to \infty$, $S(b)/b \to 0$, and $S(b)/b^{\max\{\alpha-2\sigma,-\sigma\}} \to \infty$, as required.

Note that economically this means that if the essential good is too inelastic then the gain from high commodity taxation outweigh the dead-weight losses which grow at a rate $r \in (0, 1)$. Figure 5 shows the numerical ranges of such σ.

A.5 An Optimal Exchange Rate

Partial Spheres of Exchange (PSOE): Under PSOE, it is possible to exchange luxury goods for subsistence goods at an exchange rate decided by the policy maker. We assume that the planner then uses the revenue to redistribute to the poor. The budget set for agents now becomes:

$$p_f(f - \hat{f}) + p_l(l - \hat{L}) \leq I + r(\kappa)$$
$$p_{s_!}(s_1 - \hat{S}_1) + p_{s_2}(s_2 - \hat{S}_2) + (1 + \kappa \mathbb{I}_{I \geq 0})I \leq 0.$$

Here κ represents the (exogenous) exchange rate, $r(\kappa)$ is the per-capita revenue to the poor and I is the choice as to how much you want to transfer your endowment from one sphere to the other.

Proposition 5. Suppose the planner redistributes per-capita revenue $r(\kappa)$ to the poor. Since the poor are not endowed with essentials and luxury goods, and hence do not convert across spheres, their indirect utility is

$$V_B(\kappa) = \max_{p_f(f-\hat{f})+p_l(l-\hat{L}) \leq r(\kappa)} U(\cdot),$$

where $U(\cdot)$ is given by (2). By the envelope theorem,

$$\frac{dV_B(\kappa)}{d\kappa} = \lambda_B \left(\frac{dr}{d\kappa} - \frac{dp}{d\kappa} f_B \right).$$

Landowners are endowed with essentials ($\hat{f}_A$) but do not convert across spheres and receive no transfers, so their indirect utility satisfies

$$V_A(\kappa) = \max_{p_f(f-\hat{f})+p_l(l-\hat{L}) \leq 0} U(\cdot),$$

and the envelope theorem gives

$$\frac{dV_A(\kappa)}{d\kappa} = \lambda_A \left(-\frac{dp}{d\kappa}(f_A - \hat{f}_A) \right).$$

Traders are the only agents who convert across spheres and are not endowed with essentials. Their budget constraints are

$$p_f(f - \hat{f}) + p_l(l - \hat{L}) \leq I,$$
$$p_{s_1}(s_1 - \hat{S}_1) + p_{s_2}(s_2 - \hat{S}_2) + (1 + \kappa)I \leq 0,$$

where $I > 0$ denotes equilibrium cross-sphere conversion. Let λ_C and μ_C denote the Lagrange multipliers on the essential- and luxury-sphere constraints, respectively. The first-order condition with respect to I implies

$$\lambda_C - (1 + \kappa)\mu_C = 0, \qquad \text{so that} \qquad \mu_C = \frac{\lambda_C}{1 + \kappa}.$$

Applying the envelope theorem to the trader's problem yields

$$\frac{dV_C(\kappa)}{d\kappa} = -\lambda_C \frac{dp}{d\kappa} f_C - \mu_C I = -\lambda_C \frac{dp}{d\kappa} f_C - \frac{\lambda_C}{1 + \kappa} I.$$

The planner chooses κ to maximize utilitarian welfare

$$\max_{\kappa} \; aV_A(\kappa) + bV_B(\kappa) + cV_C(\kappa).$$

The first-order condition is therefore

$$a\frac{dV_A}{d\kappa} + b\frac{dV_B}{d\kappa} + c\frac{dV_C}{d\kappa} = 0.$$

Substituting the envelope expressions gives

$$\frac{dr}{d\kappa}(b\lambda_B) - \frac{dp}{d\kappa}\Big[a\lambda_A(f_A - \hat{f}_A) + b\lambda_B f_B + c\lambda_C f_C\Big] - c\,\frac{\lambda_C}{1 + \kappa} I = 0.$$

Finally, budget balance under Partial Spheres of Exchange implies

$$r = \frac{c}{b}\,\kappa I, \qquad \frac{dr}{d\kappa} = \frac{c}{b} I + \frac{c}{b}\kappa\frac{dI}{d\kappa}.$$

Substituting this expression into the first-order condition yields the stated optimality condition. □

A.6 Complementarity

The budget constraint for agent of type i can be given by: $B(p, \tau, \kappa, R)$ is as follows:

$$p_f(f - \hat{f}) + p_l(l - \hat{L}) \leq I + \frac{R}{b}\mathbb{I}_{i=b}$$

$$\frac{p_s}{(1 + \tau\mathbb{I}_{(S-s_1)\geq 0})}(s_1 - S)\frac{p_s}{(1 + \tau\mathbb{I}_{(S-s_2)\geq 0})}(s_2 - S) + (1 + \kappa\mathbb{I}_{I\geq 0})I \leq 0.$$

A.6.1 Proof of Theorem 3

Proof of Theorem 3. Consider a small increase in the tax from $\tau = 0$ to $\tau = \varepsilon > 0$. By construction, the tax is non-distortionary in the luxury sphere for small τ: luxury–luxury trade remains efficient, and all luxury endowments are supplied to the market. Thus any welfare effect of the reform must operate through changes in food consumption.

Let f_A, f_B, f_C denote equilibrium food consumption of landlords, poor, and rich respectively. Food market clearing implies

$$f_A + bf_B + cf_C = 1,$$

so for the induced change from $\tau = 0$ to $\tau = \varepsilon$,

$$\Delta f_A + b\,\Delta f_B + c\,\Delta f_C = 0 \quad \Longrightarrow \quad c\,\Delta f_C = -b\,\Delta f_B - \Delta f_A. \tag{37}$$

Total welfare changes only through food consumption. Let λ_i denote the marginal utility of food for group $i \in \{A, B, C\}$ evaluated at $\tau = 0$. Then the first-order welfare change is

$$\Delta W = \lambda_A\,\Delta f_A + \lambda_B\,b\,\Delta f_B + \lambda_C\,c\,\Delta f_C. \tag{38}$$

Use (37) to eliminate Δf_c:

$$\begin{aligned}\Delta W &= \lambda_A\,\Delta f_A + \lambda_B\,b\,\Delta f_B + \lambda_C(-b\,\Delta f_B - \Delta f_A)\\ &= -(\lambda_C - \lambda_A)\,\Delta f_A + b(\lambda_B - \lambda_C)\,\Delta f_B.\end{aligned} \tag{39}$$

We now find the sign of Δf_A and Δf_B.

Step 1: The food price increases. Let p denote the food price faced by landlords and poor agents. For $\kappa > 0$, rich agents face the effective food price $p(1+\kappa)$. Let $x(\cdot)$ denote individual food demand at interior solutions. For sufficiently small $\tau = \varepsilon > 0$, the induced income loss for rich agents is second order, and—by quasi-linearity in luxuries—food demand is locally independent of income. Hence all first-order demand responses operate only through prices.

With total tax revenue $R(\tau;\kappa)$ rebated lump-sum to the poor, food market clearing is

$$1 = x(p) + c\,x\big(p(1+\kappa)\big) + b\,\frac{L + R(\tau;\kappa)/b}{p}. \tag{40}$$

Multiplying by p yields

$$p\,x(p) + c\,p\,x\big(p(1+\kappa)\big) + bL + R(\tau;\kappa) = p. \tag{41}$$

Differentiating (41) with respect to τ at $\tau = 0$, holding κ fixed and denoting first-order changes by $\Delta(\cdot)$, we obtain

$$\Delta p\big[1 - x(p) - c\,x\big(p(1+\kappa)\big)\big] = b\,\Delta R + p\big(\Delta f_A + c\,\Delta f_C\big). \tag{42}$$

By assumption, food demand is downward sloping, so

$$\frac{\partial x(p)}{\partial p} < 0, \qquad \frac{\partial x(p(1+\kappa))}{\partial p} < 0.$$

Moreover, $1 - x(p) - c\,x(p(1+\kappa)) > 0$, since landless agents consume a strictly positive share of food.

Suppose, for contradiction, that $\Delta p \leq 0$. Then $\Delta f_A \geq 0$ and $\Delta f_C \geq 0$, while $\Delta R > 0$ for any $\varepsilon > 0$. Hence the right-hand side of (42) is strictly positive, while the left-hand side is nonpositive, a contradiction. Therefore,

$$\Delta p > 0. \tag{43}$$

Step 2: Given price increase, landlords and traders consume less and landless consume more. Since $\partial x/\partial p < 0$, (43) implies

$$\Delta f_A < 0, \qquad \Delta f_C < 0. \tag{44}$$

Plugging (44) into the market–clearing condition (37), we obtain

$$\Delta f_B = -\frac{1}{b}\big(\Delta f_A + c\,\Delta f_C\big) > 0.$$

Step 3: Ordering of marginal utilities. At $\tau = 0$, landlords have the highest food consumption, then rich traders (who consume at most as much as landlords do), then the poor. With concave utility in food, $u''(f) < 0$, the marginal utility of food is strictly decreasing in f.

Hence, evaluated at $\tau = 0$,

$$\lambda_B > \lambda_C \geq \lambda_A. \tag{45}$$

Step 4: Welfare effect. Combining (39), (44), and (45), we have

$$\Delta W = -(\lambda_C - \lambda_A)\,\Delta f_A + b(\lambda_B - \lambda_C)\,\Delta f_B.$$

By (44), $\Delta f_A < 0$ and $\Delta f_B > 0$, and by (45), $\lambda_C - \lambda_A \geq 0$ and $\lambda_B - \lambda_C > 0$. Therefore both terms on the right-hand side are weakly positive, and the second term is strictly positive. It follows that

$$\Delta W > 0.$$

Thus a small increase in the commodity tax rate τ from zero, with all revenue rebated to the poor, strictly increases poor welfare (and aggregate welfare). This proves Theorem 3. □

A.6.2 Proof of Theorem 4

We proceed by cases, depending on whether the initial commodity tax rate $\tau > 0$ is non-distortionary or distortionary.

Case 1: τ is non-distortionary. When τ is non-distortionary, luxury–luxury trade remains efficient and all luxury endowments are fully supplied to the market. Introducing a small exchange rate $\kappa > 0$ therefore raises revenue without affecting the allocation in the luxury sphere. This revenue is rebated lump-sum to the poor and spent on essentials. By the same price-mediated argument as in Theorem 3, this strictly increases poor welfare, and as long as marginal utilities are ordered as in Theorem 3, the utilitarian objective function. Hence, the result follows immediately in this case.

Case 2: τ is distortionary. Suppose the initial policy is $(\tau^*, 0)$ with $\tau^* > \tau^{ND*}$, where

$$\tau^{ND*} := \frac{\gamma_1}{\gamma_2} - 1$$

is the maximal non-distortionary commodity tax. Without loss of generality, assume $\gamma_1 > \gamma_2 > 0$. Under $(\tau^*, 0)$, traders sell luxuries to finance essential-good consumption, and luxury–luxury trade collapses. Let p_0 denote the equilibrium food price under this policy.

Under commodity taxation alone ($\kappa = 0$), total revenue equals

$$R(\kappa = 0) = c\,\tau^*\,p_0\left[x\left(\frac{p_0\gamma_2}{\gamma_1}(1+\tau^*)\right) - L\right],$$

which is the value of essential-good purchases above subsistence financed by selling luxuries.

We now construct an alternative joint policy (τ', κ') that weakly dominates $(\tau^*, 0)$ in terms of poor welfare. Specifically, set

$$\tau' = \tau^{ND*}, \qquad 1 + \kappa' = \frac{\gamma_2}{\gamma_1}(1 + \tau^*),$$

which implies $\kappa' > 0$ because $\tau^* > \tau^{ND*}$. We refer to this regime as *commodity taxation with partial spheres of exchange* (CT with PSOE).

Under (τ', κ'), commodity taxation is non-distortionary, so luxury–luxury trade is fully efficient. By symmetry, consider a trader of type 1. Since $\gamma_1 > \gamma_2$, such a trader supplies only good s_2 to the market. The budget constraints are

$$p_f(f - \hat{f}) + p_l(l - \hat{L}) \leq I,$$
$$\frac{p_s}{1 + \tau'}(s_2 - S) + (1 + \kappa' \mathbb{1}_{\{I \geq 0\}})I \leq 0.$$

Revenue from non-distortionary commodity taxation equals

$$\frac{\tau'}{1 + \tau'}\gamma_1 cS = (\gamma_1 - \gamma_2)cS.$$

Let p_1 denote the equilibrium food price under (τ', κ'). Traders face the effective food price $p_1(1 + \kappa')$, so their food demand is $x\big(p_1(1 + \kappa')\big)$. Total revenue under the joint policy is therefore

$$R(\kappa' > 0; \tau') = c\,\kappa' p_1 \Big[x\big(p_1(1 + \kappa')\big) - L\Big] + (\gamma_1 - \gamma_2)cS.$$

We now compare prices under the two regimes.

Claim 1. It must be that $p_1 \geq p_0$.

Proof. Suppose $p_1 < p_0$. Consumptions of foodgrains under $(\tau^*, 0)$ are:

$$f_A = x(p_0) \quad f_C = x\left(\frac{p_0\gamma_2}{\gamma_1}(1 + \tau^*)\right) \quad f_B = \frac{L + \frac{R(\kappa=0)}{b}}{p_0}$$

where x(.) represent strictly decreasing demand functions. Note that our primitives assure us that demand is strictly decreasing in prices.

Total consumption under (τ', κ') is:

$$f_A = x(p_1) \quad f_C = x\left(\frac{p_1\gamma_2}{\gamma_1}(1 + \tau^*)\right) \quad f_B = \frac{L + \frac{R(\kappa>0,\tau')}{b}}{p_1}$$

Notice that by construction, if $p_1 < p_0$, the traders and landlords have higher consumption under (τ', κ'). We now show the poor do too, leading to a contradiction.

Poor consumption under $(\tau^*, 0)$ is

$$f_B(p_0) = \frac{L + \frac{R(\kappa=0)}{b}}{p_0} = \frac{bL + c\tau^* p_0 \left[x\left(\frac{p_0 \gamma_2}{\gamma_1}(1+\tau^*) \right) - L \right]}{b\, p_0}.$$

Under (τ', κ'),

$$f_B(p_1) = \frac{L + \frac{R(\kappa'>0;\tau')}{b}}{p_1} = \frac{bL + c\kappa' p_1 \left[x\big(p_1(1+\kappa')\big) - L \right] + cS(\gamma_1 - \gamma_2)}{b\, p_1}.$$

Since $p_1 < p_0$, replacing p_1 by p_0 in the denominator weakly reduces $f_B(p_1)$. Using

$$1 + \kappa' = \frac{\gamma_2}{\gamma_1}(1+\tau^*),$$

we have

$$x\left(\frac{p_0 \gamma_2}{\gamma_1}(1+\tau^*) \right) = x\big(p_0(1+\kappa')\big) =: x_K.$$

It therefore suffices to compare

$$\frac{bL + c\tau^* p_0 (x_K - L)}{p_0} \quad \text{and} \quad \frac{bL + c\kappa' p_0 (x_K - L) + cS(\gamma_1 - \gamma_2)}{p_0}.$$

Their difference equals

$$\frac{c}{p_0} \Big[(\kappa' - \tau^*) p_0 (x_K - L) + S(\gamma_1 - \gamma_2) \Big].$$

Using $\kappa' - \tau^* = (\gamma_2/\gamma_1 - 1)(1+\tau^*)$ and $\gamma_1 - \gamma_2 = -\gamma_1(\gamma_2/\gamma_1 - 1)$, this expression is non-negative if and only if

$$(1+\tau^*) p_0 \big[x_K - L \big] \leq \gamma_1 S,$$

which holds because traders cannot finance essential-good purchases beyond the value of their luxury endowment and the luxury budget constraint binds. Hence $f_B(p_1) \geq f_B(p_0)$. This contradicts market clearing, hence prices must not decrease.

□

Claim 2. If $p_1 \geq p_0$, then the landless consume weakly more under (τ', κ') than under $(\tau^*, 0)$.

Proof. Under $(\tau^*, 0)$, food-market clearing implies

$$b f_B(p_0) = 1 - x(p_0) - c\, x\left(\frac{p_0 \gamma_2}{\gamma_1}(1+\tau^*)\right).$$

Under (τ', κ'),

$$b f_B(p_1) = 1 - x(p_1) - c\, x\big(p_1(1+\kappa')\big) = 1 - x(p_1) - c\, x\left(\frac{p_1 \gamma_2}{\gamma_1}(1+\tau^*)\right),$$

where the last equality uses the definition of κ'. Since $x(\cdot)$ is strictly decreasing and $p_1 \geq p_0$, we have $f_B(p_1) \geq f_B(p_0)$.

Alongside the poor consuming weakly more, the landlords and the traders must weakly consume less because of demand functions decreasing in prices. Let the consumption change of the landlords, poor and traders be $\Delta f_A, \Delta f_B, \Delta f_C$ respectively. Let us compute the welfare difference:

$$W(f_A, f_B, f_C) - W(f_A + \Delta f_A, f_B + \Delta f_B, f_C + \Delta f_C)$$

By additive separability of utilities we can write this as:

$$[U(f_A) - U(f_A + \Delta f_A)] + b[U(f_B) - U(f_B + \Delta f_B)] + c[U(f_C) - U(f_C + \Delta f_C)]$$

By the means value theorem, there exists some $a' \in [f_A, f_A + \Delta f_A]$ such that:

$$[U(f_A + \Delta f_A) - U(f_A)] = \left.\frac{dU}{df}\right|_{a'} [\Delta f_A]$$

using a similar argument for the other types, we get

$$W(f_A, f_B, f_C) - W(f_A + \Delta f_A, f_B + \Delta f_B, f_C + \Delta f_C) = -\left.\frac{dU_A}{df}\right|_{a'} \Delta f_A - b \left.\frac{dU_B}{df}\right|_{b'} \Delta f_B - c \left.\frac{dU_C}{df}\right|_{c'} \Delta f_C$$

for some point at which derivatives are evaluated. Note that $\Delta f_A \leq 0, \Delta f_c \leq 0$ and $\Delta f_B \geq 0$. This and appealing to the fact that at all points (a', b', c'):

$$\frac{dU_B}{df} > \frac{dU_A}{df}, \frac{dU_C}{df}$$

and $\Delta f_A + b\Delta f_B + c\Delta f_C = 0$. Gives that this object is non-positive and thus welfare gain is non-negative. □

□

A.7 Proof of Erosion Effect

Proposition 6. *Let preferences be given by* (2) *with* $\sigma < 1$*,* $\sigma \neq 0$*. Fix* $c, S > 0$*. As the leisure endowment* L *tends to zero* ($L \to 0^+$)*, taxation with direct transfers strictly dominates spheres of exchange in terms of welfare for the landless laborers.*

Proof. Under the spheres-of-exchange (SOE) regime, the poor consume $f_B^{\mathrm{SOE}} = L/p^{\mathrm{SOE}}$, so their utility equals

$$u\left(\frac{L}{p^{\mathrm{SOE}}}\right) = \frac{1}{\sigma}\left(\frac{L}{p^{\mathrm{SOE}}}\right)^{\sigma}.$$

Because the landlords have interior demand for essentials, market-clearing implies the equilibrium price p^{SOE} stays bounded and strictly positive as $L \to 0$. Similarly, p^{DT} remains finite. We distinguish two cases.

Case 1: $\sigma < 0$**.**

Since $\sigma < 0$, we have

$$\lim_{L\to 0} u\left(\frac{L}{p^{\mathrm{SOE}}}\right) = \frac{1}{\sigma}\lim_{L\to 0^+}\frac{L^{\sigma}}{(p^{\mathrm{SOE}})^{\sigma}} = -\infty.$$

Now consider commodity taxation with lump-sum transfers (DT). If $c > 0$ and $S > 0$, there exists a tax rate $\tau > 0$ that generates strictly positive total revenue, and hence a strictly positive per-capita transfer $R > 0$ to the poor, independent of L. Poor consumption under DT is $f_B^{\mathrm{DT}} = (L+R)/p^{\mathrm{DT}}$, so

$$\lim_{L\to 0} u\left(\frac{L+R}{p^{\mathrm{DT}}}\right) = u\left(\frac{R}{p^{\mathrm{DT}}}\right) = \frac{1}{\sigma}\left(\frac{R}{p^{\mathrm{DT}}}\right)^{\sigma} > -\infty.$$

Thus for sufficiently small L, direct transfers strictly dominate SOE.

Case 2: $0 < \sigma < 1$**.** Under SOE,

$$\lim_{L\to 0} u\left(\frac{L}{p^{\mathrm{SOE}}}\right) = \frac{1}{\sigma}\lim_{L\to 0^+}\frac{L^{\sigma}}{(p^{\mathrm{SOE}})^{\sigma}} = 0.$$

Under direct transfers,

$$\lim_{L\to 0} u\left(\frac{L+R}{p^{\mathrm{DT}}}\right) = \frac{1}{\sigma}\left(\frac{R}{p^{\mathrm{DT}}}\right)^{\sigma} > 0.$$

Hence direct transfers again strictly dominate SOE for sufficiently small L.

Combining both cases, for all $\sigma < 1$, $\sigma \neq 0$, there exists $\bar{L} > 0$ such that for all $L < \bar{L}$, taxation with direct transfers yields strictly higher welfare for the poor than spheres of exchange. □

Remark 7. The formal proof is provided in the Appendix A.7. The above result also holds if the *marginal value* (equilibrium price) of leisure approaches zero, rather than the endowment itself.