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Consistency and social choice

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Abstract

We consider a model of preference aggregation when a single public good has to be chosen. We do not impose any restrictions on the preferences. We show the impossibility of contraction consistent (CC), anonymous and Pareto efficient social choice functions. We provide a characterization of the priority based social choice function (Priority Rule) which satisfies a weaker version of consistency called Efficient Dominance (ED). ED is a Weak Axiom of Revealed Preference (WARP) type of consistency criterion over the set of Pareto efficient alternatives. We show that the Priority Rule is the only social choice function that satisfies Pareto efficiency and Efficient Dominance.

JEL classification: D71, D72

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1 Introduction

The Arrow's theorem proves an impossibility in an unrestricted domain with Independence of Irrelevant Alternatives (IIA), Anonymity and Pareto optimality. It seems natural to assume that while choosing from a set of alternatives replacing IIA with contraction consistency (CC) would produce the same result since IIA is equivalent to CC in the individual choice theory setting (see Austen-Smith and Banks (2000) for a comprehensive survey on the individual choice theory literature). Moreover, both conditions require immunity from the removal of irrelevant alternatives. However, the fact that such an equivalence should extend to the social choice setting is not obvious since individual preferences have to be taken into account. Moreover, IIA does not consider changes in the set of alternatives over which individuals have preferences (as in Sen (1970)). In this paper, we prove the impossibility of social choice to be contraction consistent directly. In addition, we provide a characterization of a priority-based social choice rule which satisfies a weaker notion of consistency.

Contraction consistency requires that if irrelevant alternatives drop out, then the social choice function should continue to pick the same outcome. This is similar to the Chernoff's condition (Chernoff (1954)) and the α consistency introduced in Sen (1971). In other words, if the feasible set of alternatives contract in a manner which the initial outcome is still available, then the same alternative should continue to be chosen.

Contraction consistency is one of the central axioms of rationalizability as shown in Sen (1971). However, Theorem 1 of our paper shows that there are no social choice functions which satisfy contraction consistency, anonymity and Pareto efficiency. Therefore, our result points to significant challenges in the rationalizability of the social choice function and is not as predicted by individual choice theory. This is also observed in other papers in the literature.

If contraction consistency had the same bite in the social choice setting as in the individual choice setting, the social choice function chooses the highest ranked alternative from the set of Pareto efficient allocations would satisfy contraction consistency.[1] However, Theorem 1 implies that this is not true. The following example illustrates this.

Example 1 *Consider the following profile with any n voters, $\pi \in \mathcal{P}(S)^n$,*

[1] An alternative is Pareto efficient in a given profile of preferences if there is no other alternative which is strictly preferred to it by every agent.

$$\pi = \begin{pmatrix} \overbrace{b \cdots b}^{k} & \overbrace{c \cdots c}^{n-k} \\ c \cdots c & a \cdots a \\ a \cdots a & b \cdots b \end{pmatrix}.$$

Note that only b and c are Pareto efficient in the above profile since c is strictly preferred to a by each voter and the feasible set is $S = \{a, b, c\}$. The Priority Rule, f^P picks the highest ranked alternative from the set of Pareto efficient alternatives in the given profile with respect to the exogenous fixed ordering.[2] *Let the fixed ordering be $a \succ b \succ c$. Note that in the given profile, only b and c are Pareto efficient since c is strictly preferred over a by every voter. Therefore, since $b \succ c$, we have $f^P(\pi) = b$ as the social outcome. However, if the set of feasible alternative contracts to $S' = \{a, b\}$, then $f^P(\pi_{\{a,b\}}) = a$ as a is Pareto Efficient in $\{a, b\}$ and $a \succ b$. Hence, $f^P(\pi) \neq f^P(\pi_{\{a,b\}})$. This is a violation of contraction consistency.*

Therefore, the Priority rule does not satisfy contraction consistency. Theorem 1 states that there are no social choice functions which satisfy Pareto efficiency, anonymity and contraction consistency.

We first provide a basic intuition of Theorem 1. The proof of the above theorem works in multiple steps. We define the quota for a pair of alternatives as the minimum votes required at a profile (defined or restricted to the two given alternatives) for one alternative to be chosen against the other. We show that there is one pair of alternatives for which the quota is less than or equal to half the number of voters. If there is no such pair, we can construct profiles over a set of three distinct alternatives which would result in a violation of contraction consistency. The next lemma proves a monotonicity property of the social choice rule, i.e., if an alternative receives more number of votes at the top than the quota with respect to the other alternative when the only available alternatives are the two given alternatives, the fixed alternative continues to be chosen by the social choice function. The next two lemmas show that for a fixed pair of distinct alternatives a and b, the quota of a against c will be less than or equal to the quota of a against b for any other alternative c. Finally, we argue that there is no social choice function which satisfies all the given axioms by using the pair of alternatives over which the quota is less than the majority and applying the subsequent lemmas to show that a contradiction arises.

Our next theorem provides a characterization of the priority rule which picks the highest ranked alternative from the set of Pareto efficient alternatives in the given profile. We show that a social choice function is the Priority rule if and only if it

[2]We define binary relation $\succsim$ on X as an ordering if it is reflexive ($x \succsim x$ for all $x \in X$), complete ($x \succsim y$ or $y \succsim x$ for all $x \in X$), transitive ($x \succsim y$ and $y \succsim z$ implies $x \succsim z$) and antisymmetric ($x \succsim y$ and $y \succsim x$ implies $x = y$).

satisfies Pareto efficiency and Efficient Dominance (ED). ED requires that if there exists of a profile over a set S where both x and y are Pareto efficient and x is chosen, then for any other profile over any other set T in which both x and y are Pareto efficient, y must not be chosen.

Priority rule is applicable in many voting settings where the planner may have a preference over the set of candidates or alternatives. Suppose voters who are themselves candidates want to select a representative. In such a case, the set of feasible options or candidates may be a subset of all the voters. In such a case, the planner may want to choose her most preferred candidate from the set of Pareto efficient candidates in the given preference profile. Such decision processes are common in departmental committees where the Head of the department may pick her preferred candidate from the set of Pareto efficient candidates. The relevance of Theorem 2 is to highlight the unique consistency property of such a rule.

Most of the classical literature on social choice like Sen (1970) and Sen (1971) did not consider individual preferences when the feasible set of alternatives contract. The structure of the preference domain was not taken into account when the feasible set contracts. There are some papers which look at contraction consistency in social choice in restricted domains. However, most of these papers do not consider the change in voter preferences across different feasible sets. We discuss some works which do.

Dasgupta (2011) consider a stochastic model of social choice and provide necessary and sufficient conditions on the domain for equivalence between contraction consistency and the stochastic variant of the WARP. Bhattacharya (2023) considers generalized single-peaked preferences over trees and find that threshold-based rules are contraction consistent. Koray (2000) provides an impossibility result for 'self-stable' or 'self-selective' social choice functions which are unanimous and neutral. Self-stability is related to contraction consistency since the axiom requires ranking different social choice functions according to the outcomes they produce. In this context, if the set of social choice functions produce a subset of the feasible alternatives, the set of outcomes effectively 'contracts'. However, they assume neutrality of the social choice functions and different social choice functions may produce the same outcome. Due to this, our axiom is different but both produce an impossibility.[3]

Chandrasekher (2015) uses a similar notion of consistency called dynamic consistency which requires a similar check as contraction consistency and characterizes the set of anonymous, unanimous, strategy-proof and dynamically consistent social choice functions in the single-peaked domain. It shows that monotone-threshold rules which

[3] Koray and Unel (2003) and Koray and Slinko (2008) extend these results by weakening the conditions imposed on the possible set of social choice functions, i.e., the constitution.

are similar to Moulin (1980)'s generalized median voter rules are the only rules that satisfy these properties. Our paper takes this approach of defining consistency of social choice keeping in mind the restriction of preferences in the profile. Brandt and Harrenstein (2011) characterizes a notion of set-rationalizability similar to α and γ consistency of Amartya Sen (Sen (1971)) called self-stability. Our paper differs from the classical social choice approach in as in Sen (1971) and Brandt and Harrenstein (2011) which keep the preferences of individual in the background since our consistency notion requires preferences to be tracked as the set contracts.

The paper is organized as follows. Section 2 provides the notations and model. Section 3 provides the Impossibility Result and Section4 provides the characterization of the Priority Rule. Section 5 concludes.

2 Model

The set of alternatives is $X = \{a, b, c..., x, y, z...\}$ such that $|X| = m$ and the set of voters is $N = \{1, 2, ..., n\}$. Let $\mathcal{X}$ denote the set of all non-empty subsets of X. Each voter $i \in N$ has strict preference ordering P_i which is complete, asymmetric and transitive. Let $\mathcal{P}(S)$ denote the set of all strict preference orderings over S for any $S \in \mathcal{X}$. A profile $\pi \in \mathcal{P}(S)^n$ is a tuple $(P_1, P_2, ..., P_n)$ is a set of n voter preferences over S for any $S \in \mathcal{X}$. A social choice function (s.c.f.) $f : \mathcal{P}(S)^n \to S$ is a mapping from any profile π over S to $f(\pi) \in S$ for any $S \in \mathcal{X}$. Let $\pi_S = (P_1, ..., P_n)|_S$ denote the restriction of the profile $\pi \in \mathcal{P}(X)^n$ to the set S. Therefore, for any $S, T \in \mathcal{X}$ where $S \subset T$, π_S denotes the restriction of profile π_T to $S \in \mathcal{X}$ where π_S and π_T are restrictions of the profile π to S and T respectively. Let $PE(\pi_S) = \{x \in S : \nexists y \in S \text{ s.t. } yP_ix \; \forall i \in N\}$ denote the set of Pareto efficient alternatives in π_S for any $S \in \mathcal{X}$. The following two axioms are minimum requirements that any appropriate social choice function should satisfy.

Definition 1 (Pareto efficiency (PE)) *An s.c.f. is Pareto efficient if for any $S \in \mathcal{X}$ and for all $\pi \in \mathcal{P}(X)^n$, $f(\pi_S) \in PE(\pi_S)$.*

Pareto efficiency (PE) is a standard requirement for economic efficiency. For any profile over a set of alternatives S, the outcome of a social choice function must be such that no other alternative can make everyone (strictly) better-off. In other words, if there is another alternative y which an agent prefers strictly to the given outcome, there is another agent who (strictly) prefers the outcome to y.

Definition 2 (Anonymity (AN)) *An s.c.f. is Anonymous if for any $S \in \mathcal{X}$ and for all $\pi \in \mathcal{P}(X)^n$ and $\sigma : N \to N$ the following holds,*

$$f(\pi_S^\sigma) = f(\pi_S)$$

where $\pi_S^\sigma = (P_{\sigma(1)}, P_{\sigma(2)}, ..., P_{\sigma(n)})|_S$.

Anonymity requires that the names or labels of individuals do not matter for the outcome of the social choice function. The next section provides an impossibility theorem.

3 An Impossibility Result

In this section, we characterize contraction consistent social choice functions from the set of anonymous and Pareto efficient social choice functions.

Definition 3 (Contraction consistency (CC)) *An s.c.f. is contraction consistent if for any* $S, T \subseteq X$ *and for all* $\pi \in \mathcal{P}(X)^n$ *the following holds,*

$$[f(\pi_T) \in S \subset T] \implies [f(\pi_S) = f(\pi_T)].$$

Theorem 1 *Suppose* $|X| \geq 3$ *and* $|N| \geq 2$. *There is no social choice function which satisfies is* ***AN, PE*** *and* ***CC***.

Proof. Let $f(x^k, y^{n-k})$ denote the profile over two alternatives x and y, where k number of voters have xP_iy and $n-k$ number of voters have yP_ix, i.e.,

$$f(x^k, y^{n-k}) = f\begin{pmatrix} \overbrace{x \cdots x}^{k} & \overbrace{y \cdots y}^{n-k} \\ y \cdots y & x \cdots x \end{pmatrix}.$$

Let $q(x, y)$ be defined as follows:

$$q(x,y) = \arg\min_{k \in N} \left[f\begin{pmatrix} \overbrace{x \cdots x}^{k} & \overbrace{y \cdots y}^{n-k} \\ y \cdots y & x \cdots x \end{pmatrix} = x \right] \quad (1)$$

i.e. $q(x, y)$ is the minimum number of voters (any) with xP_iy for which x is chosen from a profile consisting of only x and y. By **AN**, the number $q(x, y)$ is well-defined irrespective of which individuals have the given preferences. Note that $q(x, y) \in \{1, 2, ..., n\}$ and by **AN** the order of the agents' preferences does not matter. We prove that there is no s.c.f. which satisfies the three axioms. We use a series of Lemmata to prove this.

Lemma 1 *There exists one pair of distinct alternatives* $x, y \in X$ *such that* $q(x, y) \leq \frac{n}{2}$.

Proof. We prove by contradiction. The following proof works irrespective of whether n is odd or even. Suppose $q(x, y) > \frac{n}{2}$ for all distinct $x, y \in X$. Consider any three distinct alternatives $a, b, c \in X$. Let $P = aPbPc...$, $P' = bPcPa....$ and $P'' = cP''aP''b.....$ Consider a profile $\pi \in \mathcal{P}^n$ such that (i) $k < \frac{n}{2}, q < \frac{n}{2}, n-k-q < \frac{n}{2}$ (ii) $q + n - k - q = n - k > \frac{n}{2}$ and $k + n - k - q = n - q > \frac{n}{2}$. Note that $n - k - q < \frac{n}{2}$ in (i) implies that $k + q > \frac{n}{2}$. By **CC**,

$$f(P^k, P'^q, P''^{n-k-q}) = a \implies f(a^k, c^{n-k}) = a.$$

However, since $q(a,c) > \frac{n}{2}$ and $k < \frac{n}{2}$ we have, $f(a^k, c^{n-k}) \neq a$. This is a contradiction. Similarly,

$$f(P^k, P'^q, P''^{n-k-q}) = b \implies f(b^q, a^{n-q}) = b.$$

However, since $q(b,a) > \frac{n}{2}$ and $q < \frac{n}{2}$ we have, $f(b^q, a^{n-q}) \neq b$. Therefore, this is a contradiction. Similarly,

$$f(P^k, P'^q, P''^{n-k-q}) = c \implies f(b^{k+q}, c^{n-k-q}) = c.$$

However, since $q(c,b) > \frac{n}{2}$ and $n-k-q < \frac{n}{2}$ we have, $f(b^{k+q}, c^{n-k-q}) \neq c$. Therefore, this is a contradiction. Hence, there exists one pair of distinct alternatives x, y such that $q(x,y) \leq \frac{n}{2}$. ■

Lemma 2 *$f(a^k, b^{n-k}) = a$ for all $k \geq q(a,b)$.*

Proof. If $k = q(a,b)$ then $f(a^k, b^{n-k}) = a$ by definition of $q(a,b)$. For any $k \in \{q(a,b)+1, ..., n\}$, we show that,

$$f(a^k, b^{n-k}) = f\begin{pmatrix} \overbrace{a \cdots a}^{k} & \overbrace{b \cdots b}^{n-k} \\ b \cdots b & a \cdots a \end{pmatrix} = a \text{ for all } k \in \{q(a,b)+1, ..., n\}.$$

Consider the following profile, $\pi \in \mathcal{P}(S)^n$ where $S = \{a, b, c\}$,

$$f\begin{pmatrix} \overbrace{a \cdots a}^{q(a,b)} & \overbrace{c \cdots c}^{k-q(a,b)} & \overbrace{c \cdots c}^{n-k} \\ c \cdots c & a \cdots a & b \cdots b \\ b \cdots b & b \cdots b & a \cdots a \end{pmatrix}$$

If $f(\pi) = a$, then by **CC**, $f(\pi) = f(\pi_{\{a,b\}}) = f(a^k, b^{n-k}) = a$ and our claim is true. So suppose, $f(\pi) \neq a$. By **PE**, $f(\pi) \neq b$ since cP_ib for all $i \in N$. If $f(\pi) = c$, then by **CC**, $f(\pi) = f(\pi_{\{a,c\}}) = f(a^{q(a,b)}, c^{n-q(a,b)}) = c$ $(*)$.

Consider the following profile, $\pi' \in \mathcal{P}(S)^n$,

$$f\begin{pmatrix} \overbrace{a\cdots a}^{q(a,b)} & \overbrace{b\cdots b}^{n-q(a,b)} \\ b\cdots b & c\cdots c \\ c\cdots c & a\cdots a \end{pmatrix}.$$

By **PE**, $f(\pi') \neq c$ since bP_ic for all $i \in N$. If $f(\pi') = b$, then by **CC**, $f(\pi') = f(\pi'_{\{a,b\}}) = f(a^{q(a,b)}, b^{n-q(a,b)}) = b$. This is a contradiction to the definition of $q(a,b)$ which implies that $f(\pi'_{\{a,b\}}) = f(a^{q(a,b)}, b^{n-q(a,b)}) = a$. Therefore, it must be the case that $f(\pi') = a$. By **CC**, $f(\pi') = f(\pi'_{\{a,c\}}) = f(a^{q(a,b)}, c^{n-q(a,b)}) = a$. However, by $(*)$, $f(a^{q(a,b)}, c^{n-q(a,b)}) = c$. This is a contradiction. Therefore, $f(a^k, b^{n-k}) = a$ for all $k \in \{q(a,b)+1, ..., n\}$.

■

Lemma 3 *If $q(a,b) = k$ for some $b \in X \setminus \{a\}$ then $q(a,c) \leq k$ for all $c \in X \setminus \{a\}$.*

Proof. Suppose $q(a,b) = k$ for some $a, b \in X$. We show that $q(a,c) \leq k$ for all $c \in X \setminus \{a\}$. Consider the following profile, $\pi \in \mathcal{P}(S)^n$,

$$f\begin{pmatrix} \overbrace{a\cdots a}^{k} & \overbrace{b\cdots b}^{n-k} \\ b\cdots b & c\cdots c \\ c\cdots c & a\cdots a \end{pmatrix}.$$

By **PE**, $f(\pi) \neq c$ since bP_ic for all $i \in N$. If $f(\pi) = b$, then by contraction consistency, $f(\pi) = f(\pi_{\{a,b\}}) = f(a^k, b^{n-k}) = b$. This is a contradiction to the fact that $q(a,b) = k$ which implies that $f(\pi_{\{a,b\}}) = f(a^k, b^{n-k}) = a$. Therefore, $f(\pi) = a$ which by **CC** implies that $f(\pi_{\{a,c\}}) = f(a^k, c^{n-k}) = a$. Therefore, $q(a,c) \leq k$. ■

Lemma 4 *If $q(a,b) = k$ then $q(b,c) \leq k$ for all $c \in X \setminus \{a\}$.*

Proof. Suppose $q(a,b) = k$. Consider the following profile, $\pi \in \mathcal{P}(S)^n$,

$$f\begin{pmatrix} \overbrace{b\cdots b}^{k} & \overbrace{c\cdots c}^{n-k} \\ a\cdots a & b\cdots b \\ c\cdots c & a\cdots a \end{pmatrix}.$$

By **PE**, $f(\pi) \neq a$ since bP_ia for all $i \in N$. If $f(\pi) = c$, then by **CC**, $f(\pi) = f(\pi_{\{a,c\}}) = f(a^k, c^{n-k}) = c$. This is a contradiction to Lemma 3 which states that $q(a,b) = k$ implies $q(a,c) \leq k$. This further implies that $f(\pi_{\{a,c\}}) = f(a^k, c^{n-k}) = a$. Therefore, $f(\pi) \neq c$. Therefore, $f(\pi) = b$ which by **CC** implies that $f(\pi_{\{b,c\}}) = f(b^k, c^{n-k}) = b$. Therefore, $q(b,c) \leq k$.

■

We now show that there is no s.c.f. which satisfies the three axioms. By Lemma 1 there is one pair of alternatives a, b with $q(a,b) = k \leq \frac{n}{2}$. Applying Lemma 4 to b and c we get $f(b^k, c^{n-k}) = b$ which implies that $q(b,c) \leq k$. However, the same arguments in Lemma 4 can be repeated by reversing the roles of b and c. Therefore, applying Lemma 4 to c and b we get $f(c^k, b^{n-k}) = c$. Note that $n - k \geq k$ since $k \leq \frac{n}{2}$. By Lemma 2 that $q(b,c) \leq k$ implies $f(b^k, c^{n-k}) = f(b^{n-k}, c^k) = b$. By **AN**, this implies that $f(c^k, b^{n-k}) = b$. This is a contradiction.

Independence of the axioms: The following rule satisfies **PE** and **CC** but not **AN**: **dictatorial rule:** An s.c.f. is dictatorial w.r.t. $\bar{i}$ if for any $S \in \mathcal{X}$ and for all $\pi \in \mathcal{P}(X)^n$, $f(\pi_S) = \max_{P_{\bar{i}}}(\pi_S)$, i.e., the maximal alternative from S according to $P_{\bar{i}}$ is chosen.

The following rule satisfies **CC** and **AN** but does not satisfy **PE**: **Priority Rule* (PR*):** An s.c.f. f^{PR^*} is **PR*** if there exists an ordering $\succsim$ on X such that for any $S \in \mathcal{X}$ and for all $\pi \in \mathcal{P}(X)^n$, $f^{PR^*}(\pi_S) = \max_{\succ}(S)$.[4]

■

Theorem 1 shows that there is no s.c.f. which satisfies contraction consistency, anonymity and Pareto efficiency. We provide a brief sketch of the proof. We first consider 'binary' profiles, i.e., when only two alternatives are present. For any pair of alternatives, we compute the minimum votes or quota (the fact that it is a quota is proved in the third lemma) required for an alternative to be chosen against another alternative when it is in the top ranked set of alternatives. The impossibility of an s.c.f. is proved in multiple lemmata. We first show that there exists one pair of alternatives x and y for which the quota is less than or equal to $\frac{n}{2}$. We prove this by contradiction: if no such pair exists, then by defining a specific profile consisting of three alternatives, (a, b, c), we obtain a contradiction to contraction consistency over binary profiles over the three pairs $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$. Lemma 2 proves that the quota indeed operates as a threshold value, if x appears more than $q(x, y)$ number of times then that alternative is chosen. The final step uses this to show that contraction consistency is violated for any given set of thresholds.

Therefore, 1 indicates that contraction consistency is a strong restriction to be imposed on anonymous and Pareto efficient social choice functions. In the next section, we drop contraction consistency and introduce an axiom, Efficient Dominance (ED), which is similar to the Weak Axiom of Revealed Preference (WARP) operating on the set of Pareto efficient alternatives. This is satisfied by a priority-based social choice function.

[4] The priority rule, **PR**, defined in the next section satisfies **AN** and **PE** but not **CC**.

4 A Characterization of the Priority Rule

We define a priority-based rule which is relevant in many voting situations where the planner has a fixed ordering over the set of candidates or alternatives.

Definition 4 (Priority Rule (PR)) *An s.c.f. is a* ***PR*** *if there exists an ordering* $\succ$ *on* X *such that for any* $S \in \mathcal{X}$ *and for all* $\pi \in \mathcal{P}(X)^n$ *the following holds,*

$$f^P(\pi_S) = \max_{\succ}(PE(\pi_S))$$

i.e. the priority rule picks the highest ranked Pareto efficient element in the profile π_S *for any* $S \in \mathcal{X}$. *Note that the ordering* $\succ$ *is exogenous and does not depend on the profile.*

PR can be applied to any voting situation with a planner or principal. Consider a setup where there are multiple agents (who may also be candidates) and a single principal. The representative agent has to be assigned by the principal. **PE** in this setting can be interpreted as a form of group rationality of the agents. The principal has an exogenous ordering over the set of agents or candidates who report their preferences over the set of agents. The principal can then use the Priority Rule to choose the representative agent.

Example 2 *Consider the following profile,* $\pi_S \in \mathcal{P}(S)^n$,

$$\pi_S = \begin{pmatrix} \overbrace{b \cdots b}^{k} & \overbrace{c \cdots c}^{n-k} \\ c \cdots c & a \cdots a \\ a \cdots a & b \cdots b \end{pmatrix}.$$

where $S = \{a, b, c\}$, $k \in \{1, \cdots, n-1\}$. *Consider the* ***PR*** *with respect to the exogenous fixed ordering* $a \succ b \succ c$. *Note that in the given profile, only* b *and* c *are Pareto efficient since* c *is strictly preferred over* a *by every voter. Therefore, since* $b \succ c$, *we have* $f^P(\pi_S) = b$ *as the social outcome. However, if the set of feasible alternative contracts to* $S' = \{a, b\}$, *then* $f^P(\pi_{\{a,b\}}) = a$ *as* a *is Pareto Efficient in* $\{a, b\}$ *and* $a \succ b$. *Hence,* $f^P(\pi_S) \neq f^P(\pi_{\{a,b\}})$. *This is a violation of contraction consistency.*[5]

The following axiom in addition to **PE** is used to characterize **PR**.

Definition 5 (Efficient Dominance (ED):) *If* $\exists S \in \mathcal{X}, \exists \pi_S \in \mathcal{P}(S)^n$ *such that* $x, y \in PE(\pi_S) \subseteq S$, $x \neq y$, *and* $f(\pi_s) = x$, *then* $\forall T \in \mathcal{X}, \forall \pi'_T \in \mathcal{P}(T)^n$ *such that* $x, y \in PE(\pi'_T) \subseteq T$, *we have* $y \neq f(\pi'_T)$.

Efficient Dominance (**ED**) requires that, if for a profile over a set S where both x

[5] In terms of the implications of Theorem 1, the thresholds are $q(a, b) = 1, q(b, c) = 1$ and $q(a, c) = 1$.

and y are Pareto efficient and x is chosen, then for any other profile over any set T in which both x and y are Pareto efficient, y must not be chosen. This is similar to Weak Axiom of Revealed Preference (WARP), used in the individual choice literature (Samuelson (2024), Gale (1960)), which requires if an alternative x is chosen when another alternative y is available then if y is chosen then it must be true that x is not available. So, **ED** is similar to WARP operating over the set of Pareto efficient alternatives.

If an alternative is chosen in a larger set, it will still be **PE** in subsets. The problem arises (shown in Example 2) when alternatives which were not previously **PE** but were present in the original set (and not chosen) become PE in the subset. **ED** allows such reversals but contraction consistency does not. **ED** implies that if new elements are chosen when the set contracts, they must not be Pareto efficient in the original set.

Theorem 2 *An s.c.f. f is a Priority Rule (**PR**) if and only if f satisfies **PE** and **ED**.*

Proof. *(If-part)* Let f satisfy **PE** and **ED**. We define a binary relation $\succ$ over $\mathbf{X}$ first. Take any $x, y \in \mathbf{X}$. Then we define:

$$x \succ y \iff \exists \pi_{\{x,y\}} \in \mathcal{P}(\{x,y\})^n \text{ such that } f(\pi_{\{x,y\}}) = x \text{ and } y \in PE(\pi_{\{x,y\}})$$

We claim that $\succ$ is an ordering. For reflexivity[6], take $\{x, x\} = \{x\}$, where $x \in X$. For any preference profile, $\pi_{\{x,x\}} = \pi_{\{x\}} \in \mathcal{P}(\{x,x\})^n = \mathcal{P}(\{x\})^n$, $x \in PE(\pi_{\{x,x\}}) = PE(\pi_{\{x\}})$ and $f(\pi_{\{x,x\}}) = f(\pi_{\{x\}}) = x$. Thus $x \succ x$. For completeness of $\succ$, consider any $x, y \in \mathbf{X}$, $x \neq y$[7] and the set $\{x, y\} \in \mathcal{X}$. By Unrestricted Domain, $\exists \pi_{\{x,y\}} \in \mathcal{P}(\{x,y\})^n$ such that $x, y \in PE(\pi_{\{x,y\}})$. Since f is single-valued, either $f(\pi_{\{x,y\}}) = x$ (which would imply $x \succ y$) or $f(\pi_{\{x,y\}}) = y$ (which would imply $y \succ x$). Thus, $\succ$ is complete. For antisymmetry, let $x, y \in \mathbf{X}$ such that $x \succ y$ and $y \succ x$. If $x = y$, $\succ$ is antisymmetric. So, suppose $x \neq y$. Now $x \succ y$ implies:

$$\exists \pi'_{\{x,y\}} \in \mathcal{P}(\{x,y\})^n \text{ such that } f(\pi'_{\{x,y\}}) = x \text{ and } y \in PE(\pi'_{\{x,y\}}).$$

By **PE**, $x \in PE(\pi'_{\{x,y\}})$ as $f(\pi'_{\{x,y\}}) = x$. Thus, by **ED**, keeping $\{x, y\}$ fixed,

$$\forall \pi_{\{x,y\}} \in \mathcal{P}(\{x,y\})^n \text{ such that } x, y \in PE(\pi_{\{x,y\}}) \subseteq \{x,y\}, \text{ we have } y \neq f(\pi_{\{x,y\}}) \tag{2}$$

[6]For the purposes,of this proof, we can take $\mathcal{P}(\mathcal{S})$ as the set of all orderings (reflexive, complete, transitive, antisymmetric binary relation) over $S \in \mathcal{X}$. This is without loss of generality.

[7]We have already shown the case where $x = y$.

But $y \succ x$ implies:

$$\exists \pi''_{\{x,y\}} \in \mathcal{P}(\{x,y\})^n \text{ such that } f(\pi''_{\{x,y\}}) = y \text{ and } x \in PE(\pi''_{\{x,y\}}).$$

By noting that $y \in PE(\pi''_{\{x,y\}})$ (as $f(\pi''_{\{x,y\}}) = y$ and f satisfies PE), we arrive at a contradiction to (2) as $x \neq y$. Thus, $\succ$ is antisymmetric. For transitivity, consider $x, y, z \in \mathbf{X}$. Let $x \succ y$ and $y \succ z$. If any of x, y, z are not distinct, we have $x \succ z$. So suppose that x, y, z are all distinct. We have to show $x \succ z$. Suppose not. Then $x \not\succ z$. By completeness of $\succ$, we get $z \succ x$. Now, we have the following set of implications:

$$x \succ y \implies \exists \pi'_{\{x,y\}} \in \mathcal{P}(\{x,y\})^n \text{ such that } f(\pi'_{\{x,y\}}) = x \text{ and } y \in PE(\pi'_{\{x,y\}}) \quad (3)$$

$$y \succ z \implies \exists \pi''_{\{y,z\}} \in \mathcal{P}(\{y,z\})^n \text{ such that } f(\pi''_{\{y,z\}}) = y \text{ and } z \in PE(\pi''_{\{y,z\}}) \quad (4)$$

$$z \succ x \implies \exists \pi'''_{\{x,z\}} \in \mathcal{P}(\{x,z\})^n \text{ such that } f(\pi'''_{\{x,z\}}) = z \text{ and } x \in PE(\pi'''_{\{x,z\}}) \quad (5)$$

Now, consider $\{x, y, z\} \in \mathcal{X}$. By Unrestricted Domain, $\exists \pi_{\{x,y,z\}} \in \mathcal{P}(\{x,y,z\})^n$ such that $x, y, z \in PE(\pi_{\{x,y,z\}})$. By (3) and **ED**, $f(\pi_{\{x,y,z\}}) \neq y$. By (4) and **ED**, $f(\pi_{\{x,y,z\}}) \neq z$. By (5) and **ED**, $f(\pi_{\{x,y,z\}}) \neq x$. Thus, $f(\pi_{\{x,y,z\}}) = \emptyset$. This contradicts the fact that f is non-empty and single-valued. Thus, $x \succ z$ and $\succ$ is transitive. This shows that $\succ$ is an ordering indeed.

Now, we show that f is a Priority Rule (**PR**) with $\succ$ as the desired ordering. For that, we need to show that $\forall S \in \mathcal{X}, \forall \pi_S \in \mathcal{P}(S)^n$:

$$f(\pi_S) = \max_{\succ}(PE(\pi_S)).$$

The above is equivalent to:

$$f(\pi_S) \succ x, \forall x \in PE(\pi_S)$$

So, we take any $S \in \mathcal{X}$ and any $\pi_S \in \mathcal{P}(S)^n$. If $x = f(\pi_S)$, then by reflexivity of $\succ$, we get $f(\pi_S) \succ x$. So, now let $x \in PE(\pi_S)$ such that $x \neq f(\pi_S)$. We first show that $x, f(\pi_S) \in PE(\pi_{\{f(\pi_S),x\}})$. Note, $x \in PE(\pi_S) \implies \exists i \in \{1, \cdots, n\}$ such that $xP_i f(\pi_S)$, where P_i is the preference of the ith individual according to the profile π_S. Thus, $\forall z \in \{x, f(\pi_S)\}$, $\exists i \in \{1, \cdots, n\}$ such that $xP_i z$ with P_i being the preference of the ith individual according to the profile $\pi_{\{x,f(\pi_S)\}}$ (where $\pi_{\{x,f(\pi_S)\}}$ is the restriction of π_S to $\{x, f(\pi_S)\}$). Hence, $x \in PE(\pi_{\{f(\pi_S),x\}})$. Similarly, since f satisfies **PE**, we have $f(\pi_S) \in PE(\pi_S)$. We can repeat the same arguments as above to show that $f(\pi_S) \in PE(\pi_{\{f(\pi_S),x\}})$. Take $T = \{f(\pi_S), x\}$. By **ED**, since $x \in PE(\pi_S)$, $x \neq f(\pi_S)$, $f(\pi_S), x \in PE(\pi_{\{f(\pi_S),x\}})$, we have $x \neq f(\pi_{\{f(\pi_S),x\}})$. Since f is single-valued, we have:

$$f(\pi_{\{f(\pi_S),x\}}) = f(\pi_S).$$

Therefore, $f(\pi_S) \succ x, \forall x \in PE(\pi_S)$. This completes the proof of the *If-part* of the theorem.

(*Only if- part*): Let f^P be a **PR**. Then, there exists an ordering such that $\forall S \in \mathcal{X}, \forall \pi_S \in \mathcal{P}(S)^n$:

$$f^P(\pi_S) = \max_{\succ}(PE(\pi_S))$$

Claim 1 *f^P satisfies **PE**.*

Proof. Let $x \in S$ such that $\exists y \in S \setminus \{x\}$ with $yP_ix, \forall i \in \{1, \cdots, n\}$. Then $x \notin PE(\pi_S)$. Thus, by **PR**, $f^P(\pi_S) \neq x$. Therefore, f^P satisfies **PE**. ■

Claim 2 *f^P satisfies **ED**.*

Proof. Let $S \in \mathcal{X}$ and $\pi_S \in \mathcal{P}(S)^n$ such that $x, y \in PE(\pi_S) \subseteq S$, $x \neq y$, and $f^P(\pi_s) = x$. Take any $T \in \mathcal{X}$ and any $\pi'_T \in \mathcal{P}(T)^n$ such that $x, y \in PE(\pi'_T) \subseteq T$. We have to show that $y \neq f^P(\pi'_T)$. Suppose not. Then, by the single-valued property of f, we get $y = f^P(\pi'_T)$. We note that $x, y \in PE(\pi_S) \subseteq S$ and $f^P(\pi_s) = x$ imply, by **PR**, $x \succ y$. But $x, y \in PE(\pi'_T) \subseteq T$ and $y = f^P(\pi'_T)$ imply, by **PR**, $y \succ x$. By anti-symmetry of $\succ$, we get $x = y$. This contradicts the fact that x and y are distinct. Thus, $y \neq f^P(\pi'_T)$. This shows that f satisfies **ED**. ■

By Claim 1, and Claim 2, the proof of the *Only-if* part of the theorem is complete.

Independence of the Axioms: PR* does not pick Pareto efficient alternatives but it satisfies **ED**. In example 2 the rule **PR*** w.r.t. $a \succ b \succ c$, $f^{PR^*}(\pi_S) = a$ and $f^{PR^*}(\pi_{\{a,b\}}) = a$. Therefore, **ED** is satisfied but not **PE**.

The dictatorial rule is Pareto efficient since $\max_{P_{\bar{i}}}(\pi_S) \in PE(\pi_S)$. However, it does not satisfy **ED** since the ordering is not fixed. For example, suppose $\max_{P_{\bar{i}}}(\pi_S) = a$ with $a, b \in PE(\pi_S)$ while $\max_{P_{\bar{i}}}(\pi_T) = b$ with $a, b \in PE(\pi_T)$. Then, $f^{\bar{i}}(\pi_S) = a$ while $f^{\bar{i}}(\pi_T) = b$. This violates **ED**. ■

5 Conclusion

We show the impossibility of contraction consistent, anonymous and Pareto efficient social choice functions. We provide a characterization of the priority based social choice function which satisfies an alternative version of consistency which accounts for Pareto efficiency of alternatives in the given profile. This rule has applications for voting situations where a representative agent or candidate has to be chosen and the planner has a fixed priority over the set of candidates. Our framework applies to settings where the set of candidates or alternatives may not be fixed. Future work can explore similar notions of consistency which account for voter preferences when the set of alternatives change.

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