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Abstract

We consider an attribute-based model of stochastic choice with variable attention to different attributes. We characterize *stochastic choice rules with attributes and limited attention* (SCRALA). Under SCRALA, the probability with which an alternative (say, x) is chosen is the product of the probabilities with which attention is drawn by the attributes where x is ranked highest and the (weighted) probability with which attention is not drawn by the attributes under which x is not the highest ranked. Our results are characterized by axioms defined on observable choice data and all the attention parameters are uniquely identified.

JEL classification: D00, D01

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1 Introduction

Attributes often help in making choices as rationales to what may be relevant to the decision maker (Manzini and Mariotti (2007)). The consumer may also choose according to the salience of various attributes. However, in many cases, the decision maker (DM) may not pay full attention to each attribute. For example, an individual who wants to buy a car may be more focused on buying a sports car than a car with extra airbags. Her varying attention to the various attributes may affect how she chooses the alternative. In this paper, we characterize a choice rule that takes into account the role played by different attributes in drawing the DM's attention.

We provide some examples from consumer behaviour where attention paid to various attributes plays an important role:

(i) Consider a consumer who wants to buy a box of breakfast cereal. She may be more focused on nutritional attributes and less on the type of fruit that comes with it. The difference in attention probabilities to the two attributes may affect the choice probability of a specific cereal box. In our model, we assume that the attention paid to the attributes is independent. [1]

(ii) Consider a consumer who wants to buy a smartphone. She may pay more attention to buying a phone based on screen size instead of the number of megapixels in its camera. These two attributes are independent so the attention paid to them is assumed to be independent. We also assume that the DM is generally attentive, i.e., the attention paid to any attribute is greater than or equal to half.

We assume that choice probabilities are observable for any given subset of the set of alternatives and any subset of the set of attributes. This is a reasonable assumption since in most online and offline markets the planner can observe the set of available alternatives and their attributes. In fact, in most online shopping platforms the alternatives can be displayed on the basis of attributes as "filters". Our choice rule intends to capture the effect of varying attention to different attributes.

We characterize *Stochastic Choice Rule with Attributes and Limited Attention (SCRALA)* according to which the probability of an alternative x is the product of the probabilities with which the DM pays attention to the attributes under which x is ranked highest and the weighted probability with which attention is not drawn by those attributes under which x is not the highest ranked. The ranking over alternatives is an antisymmetric ordering. We provide an example of the rule with three alternatives and two attributes.

[1] Bialkova and van Trijp (2010) identifies the factors affecting the attention paid to different nutritional labels in an experimental setting.

Example 1 Suppose $X = \{x, y, z\}$ and $A = \{a_1, a_2\}$. Let the two orderings with respect to the attributes a_1 and a_2 be $x \succ_{a_1} y \succ_{a_1} z$ and $y \succ_{a_2} x \succ_{a_2} z$ respectively. Let $\gamma(a_1) = 0.51$ and $\gamma(a_2) = 0.8$ be the two attention probabilities of the attributes.[2] The choice probabilities of the three alternatives from the set $S = \{x, y, z\}$ according to SCRALA, p^S are as follows:

(i) $p^S(x, S, A) = \gamma(a_1)\left(\frac{1-\gamma(a_2)}{|S|-1}\right) = 0.51\left(\frac{1-0.8}{3-1}\right) = 0.51(0.1) = 0.051.$

(ii) $p^S(y, S, A) = \gamma(a_2)\left(\frac{1-\gamma(a_1)}{|S|-1}\right) = 0.8\left(\frac{1-0.51}{3-1}\right) = 0.8(0.245) = 0.196$

(iii) $p^S(z, S, A) = \left(\frac{1-\gamma(a_2)}{|S|-1}\right)\left(\frac{1-\gamma(a_1)}{|S|-1}\right) = \left(\frac{1-0.8}{3-1}\right)\left(\frac{1-0.51}{3-1}\right) = 0.0245.$

Note that the choice probability of alternative y is the highest since the attention paid to the attribute (a_2) in which it is ranked first is much higher than the attention paid to the attribute (a_1) where it is not (0.8 as compared to 0.51). Moreover, the probability of ignoring the attribute a_1 in which it is not ranked first is also relatively high (0.49). Also note that the choice probability of the default alternative is the residual probability given by $p^S(x^*, S, A) = 1 - 0.0245 - 0.196 - 0.051 = 0.7285$.

An observation from the above example is the high probability of the default alternative. An interpretation of this property in the example is the fact that DM pays attention to both attributes which have different top-preferred alternatives. This 'confuses' the DM since she does not choose according to the relative positions of x and y which are similar across the two attributes. Even though she pays higher attention to the second attribute the choice probability of y is discounted by the probability of not paying attention to the first attribute (weighted by $\frac{1}{2}$) which is high (0.49). This drags down the choice probability of alternative y in the set resulting in the DM choosing the default.

We use seven axioms to characterize SCRALA. We describe them briefly. The first axiom is *Independence (IN)* which is a standard one in many works on stochastic choice. IN requires that the probability with which an alternative is chosen from a given set of alternatives S and attribute set $(A \cup B)$ is the product of the probabilities of that alternative from the same set of alternatives over two sets of attributes A and B. Therefore, this captures the fact that the probability of choosing an alternative across two set of attributes is independent.

The second axiom is *Full Support (FS)* which states that the sum of choice probabilities for a given set of alternatives S (consisting of two or more elements) when only a single attribute is considered must add up to one (the default alternative is

[2]Note that the attention paid to attributes need not sum to one since they are assumed to be independent.

chosen with probability zero). This implies that when there is only one attribute the DM distributes the choice probabilities over alternatives in the set of alternatives and does not give any consideration to the default alternative.

The third axiom is *Invariance to Singletons (IS)* which states that the probability of choosing any alternative x from the singleton set $\{x\}$ under a singleton attribute set $\{a\}$ must be the same as the choice probability of any other alternative y from the singleton set $\{y\}$ under the same singleton attribute set. This implies that the choice probability of any alternative is dependent only on the attribute that is being considered when the set of alternatives and the set of attributes are both singleton sets.

The fourth axiom is *Uniformity of Inferior Alternatives (UIA)* which states that for any set of alternatives S and attribute set A if there exists a triple of alternatives x, y, z which belong to S such that the choice probability of x from the set $\{x, y\}$ under the attribute set $\{a\}$ is strictly greater than that of y from $\{x, y\}$ under the same attribute, and if the choice probability of x from $\{x, z\}$ under the attribute set $\{a\}$ is strictly greater than that of z from $\{x, z\}$ under the same attribute then the choice probability of the dominated alternatives i.e. y and z must be the same from S under a single attribute set $\{a\}$.

The fifth axiom is *Dominance (DOM)* which states that if an alternative, x, is chosen with a strictly higher probability than y from the binary set $\{x, y\}$ given a single attribute a for any such y which belongs to the set $S \setminus \{x\}$, then the choice probability of x is unchanged when any such y is removed from the set S.

The sixth axiom is *Binary Difference (BD)* which states that the probability of x from $\{x, y\}$ under a single attribute a must not be equal to the probability of y from $\{x, y\}$ under the same attribute. This implies that the DM can differentiate between two alternatives from a binary set when only one attribute is considered.

The seventh axiom is *Stochastic Transitivity (ST)* which is the probabilistic version of the standard transitivity axiom defined for deterministic choice functions: If x has a strictly greater choice probability than y from $\{x, y\}$ under the attribute a and if y has a strictly greater choice probability than z from $\{y, z\}$ under the attribute a, then x must have a strictly greater choice probability than z from $\{x, z\}$ under attribute a. This is an axiom that has been used in the literature on stochastic choice (see Fishburn (1973) and Manzini and Mariotti (2014).

Our main result states that a stochastic choice rule satisfies the above seven axioms if and only if it is a SCRALA. The proof of the result proceeds from binary sets in singleton attributes. The preference ordering over the set of alternatives is first derived for an attribute. We show that preference ordering is complete, transitive, and antisymmetric, where the latter is proved using BD. The IN axiom is used to derive the choice probability from a bigger set of attributes keeping the set of alternatives fixed. FS is used to derive the expression that captures the probability of not considering attributes where the alternative is not top-ranked. DOM is used to obtain the invariance of the choice probability of an alternative when removing dominated alternatives under a singleton attribute set. IN is used again to obtain the main expression of the definition of SCRALA.

We show Section 4.1 that all the parameters in SCRALA are uniquely identifiable from the choice data. BD plays an important role in this by ensuring the difference in the choice probabilities from binary sets. This leads to the attention parameters being distinct for a given choice data since no two data sets would generate the same attention parameters and preference orderings. In the next Section 4.2 we show that SCRALA satisfies a well-known property *Regularity* with respect to alternatives (for a fixed set of attributes), and also satisfies regularity with respect to attributes (for a fixed set of alternatives). The former requires that the choice probability of an alternative from a subset of *alternatives* $S \subseteq T$ be weakly greater than from T when the set of attributes is fixed. The latter requires that the choice probability of an alternative from a subset of attributes $A \subseteq B$ be weakly greater than that from B when the set of alternatives is fixed. This is the standard regularity axiom defined over alternatives as in Fudenberg et al. (2015), McFadden and Richter (1990) and Aguiar and Kimya (2019). A violation of regularity can also be captured when the set of attributes and the set of alternatives are allowed to vary. A justification for this is the Attraction Effect as introduced in Huber et al. (1982) which causes a reversal in the choice probabilities.

We show in 4.4 that our framework can also be applied to a social choice setting. The set of alternatives can be seen as the set of candidates and the set of attributes can be interpreted as the set of voters. In such a setting, SCRALA can be interpreted as assigning consideration probabilities to the voters (analogous to attention probabilities over attributes) and the choice probability of a candidate is the product of the probability with which the voters who rank that candidate the highest are considered and the (weighted) probability with which the voters who do not rank that candidate the highest are not considered. Therefore, Theorem 1 can also be applied to this setting with different voters being considered with limited attention.

SCRALA can also be interpreted as a Random Utility Model (RUM) as studied in Aguiar et al. (2023) where the attributes are 'different states of the world' and the attention probabilities over the set of attributes are the probability distribution over the different states of the world. The orderings over the set of alternatives for each attribute can be interpreted as the different preference orderings in different states of the world.

Many papers have studied the salience of attributes in stochastic individual choice. Bordalo et al. (2012) models the salience of differences in expected payoffs in a model of risk, Bordalo et al. (2013) and Bordalo et al. (2020) model the role of memory in decision-making which explains many behavioural biases including inattention to hidden product attributes. Manzini and Mariotti (2012) shows that a choice rule which takes into account different categories before choosing an alternative satisfies weak WARP in a deterministic setting. The choice rule characterized in our setting can violate weak WARP and can even violate independence of irrelevant alternatives.

We contribute to the literature on Multiple Attributes (or Criteria) based Decision Making, Limited Attention, and Stochastic Choice. Lancaster (1966) introduced a model with multiple attributes being associated with each alternative. There are papers which study the case where the criteria can be interpreted as multiple selves or rationales, e.g. Kalai et al. (2002), Manzini and Mariotti (2007), Cherepanov et al. (2013), and Hara et al. (2019). In the limited attention literature, Masatlioglu et al. (2012) provides conditions for revealed preference, while Lleras et al. (2017) provides a choice theoretic foundation of limited attention and consideration sets. A good starting point for the literature on stochastic choice is Luce (1959).

Manzini and Mariotti (2014) is closely related to our paper due to the similarity in the formulation of the stochastic choice rule. However, they model limited attention over different alternatives and the attention parameters are defined for alternatives. In our paper, a DM uses consideration probabilities for different attributes and the set of alternatives can vary as well. Another closely related paper to ours is Bhattacharya et al. (2021). They use a model concerning frames with a different Stochastic Choice Rule. Our Stochastic Choice Rule uses a different partitioning of the attribute sets with respect to each alternative. Due to this, the formulation of our rule, SCRALA, takes a functional form and the distribution of the choice probabilities to 'dominated' alternatives is different from the one in both these papers. In Bhattacharya et al. (2021), frames do not provide any additional information that helps the DM in the rational evaluation of alternatives (as in Salant and Rubinstein (2008)). In our model, the attributes are informative and help the DM in the rational evaluation of alternatives. Also, we make an additional reasonable assumption that each relevant attribute is more likely to be considered than not. A limitation of our model is that it requires a larger observable data set than these papers, which requires that choice probabilities for the alternatives are known for every set of alternatives and every given set of attributes.

Gul et al. (2014) uses attributes in a stochastic choice setting. Their axioms and the structure of the choice rule are very different as compared to ours. They seek to explain the problem of duplicates infested in the Luce Rule (Luce (1959)). An interesting model is provided by Honda (2021) where stochastic choices are made when each alternative can be associated with a category. The stochastic choice rule that is used has two terms (in a product form). The first term is representative of the probability of the categories in which there are better alternatives (than the said alternative) not being considered. The second term is the probability of consideration of the category containing the said alternative multiplied by its Luce weight with respect to the other alternatives in the same category. The weight (on the probability of not considering the attributes in which there are better alternatives, in contrast) in our paper just depends on the cardinality of the menu (of alternatives) and the cardinality of the attributes (in which the said alternative is dominated) set.

Cattaneo et al. (2020) provides a model where attention is both limited and random. They have an axiom that states that the probability of an alternative being considered weakly decreases as the menu of alternatives expands. Our model on the other hand satisfies a different but related form of monotonicity: *Regularity in Attributes*. The situation is similar: If no new alternatives are added but the relevant attribute set expands, there is a case of diminishing attention. Hence, each alternative is less likely to be considered (according to their model) and its choice probability also decreases.

Kovach and Tserenjigmid (2022) is another stochastic choice model with limited attention where some alternatives have more attention associated with them; which are known as the Focal Alternatives. Kovach and Suleymanov (2023) presents a model where reference points direct attention and show how stochastic choices are made when the attention is reference-dependent and random. Ahumada and Ülkü (2018) explains a model where a DM uses the Luce Rule on a consideration set. Kimya (2018) gives a detailed model of consideration set formation in the presence of attributes and limited attention. However, his model is based on a deterministic choice setup with thresholds unlike ours. Also, for various forms of Stochastic Transitivity see Fishburn (1973).

The paper is divided into sections. Section 2 describes the Model, while section 3 provides the axioms required for the main result. Section 4 provides the results and properties satisfied by SCRALA. An application to social choice theory is provided in Section 4.4. Section 5 concludes while the Appendix provides proof of the independence of the axioms.

2 Model

Let $\mathbf{X}$ be the set of all alternatives and $\mathcal{X}$ be the set of all finite non-empty subsets of $\mathbf{X}$. Let $\mathbf{A}$ be the set of all attributes and $\mathcal{A}$ be the set of all finite non-empty subsets of $\mathbf{A}$. We define stochastic choice rules which provide the choice probability of an alternative for every $S \in \mathcal{X}$ and $A \in \mathcal{A}$.

Definition 1 (Stochastic Choice Rule (SCR)) *An SCR, $p : \mathbf{X} \times \mathcal{X} \times \mathcal{A} \to [0, 1]$ for any $x \in \mathbf{X}, S \in \mathcal{X}$ and $A \in \mathcal{A}$ provides $p(x, S, A)$ which is the probability that alternative $x \in \mathbf{X}$ is chosen from a menu $S \in \mathcal{X}$ and the set of relevant attributes $A \in \mathcal{A}$.*

We denote a default alternative as x^* which can also be interpreted as not choosing any alternative or any other fixed alternative that is pre-determined.

The choice probabilities over the set of alternatives and the default alternative sum to one, i.e., $\sum_{x \in S} p(x, S, A) + p(x^*, S, A) = 1$, where $x \in \mathbf{X}$, $S \in \mathcal{X}$, and $A \in \mathcal{A}$. Note that $p(x, S, A) = 0$ if $x \notin S$.

Before we define the choice rule, we introduce some notations. Suppose that $\succeq_a$ is a linear ordering over X according to the attribute $a \in A$.[3] This is also the preference of the DM on the given attribute $a \in A$. We say that an alternative x *dominates* another alternative y via an attribute a if $x \succeq_a y$. For any $A \in \mathcal{A}$, $S \in \mathcal{X}$ and $x \in S$, the set

$$A_x(S) = \{a \in A : x \succeq_a y, \forall y \in S\}$$

denote the attributes in which x dominates all the alternatives in S and let

$$A_x^c(S) = \{a \in A : \exists y \in S \backslash \{x\}, y \succeq_a x\}$$

be the set of alternatives where x is dominated by some alternative $y \in S \backslash \{x\}$. We now define *Stochastic Choice Rule with Attributes and Limited Attention* (*SCRALA*).

Definition 2 (SCRALA) *An SCR, p^S is a SCRALA if for all $A \in \mathcal{A}$ there exists a linear ordering $\succeq_a$ on X and attention parameters $\gamma(a) \in (\frac{1}{2}, 1]$ for all $a \in A$ such that for any $S \in \mathcal{X}$ and $x \in S$,*

$$p^S(x, S, A) = \begin{cases} \prod_{a \in A_x(S)} \gamma(a) \dfrac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}, & \textit{if } A_x(S) \neq \emptyset, A_x^c(S) \neq \emptyset; \\ \prod_{a \in A_x(S)} \gamma(a), & \textit{if } A_x(S) = A; \\ \dfrac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}, & \textit{if } A_x^c(S) = A. \end{cases}$$

[3]We say that R is a linear ordering if it is (i) complete: xRy or yRx for all x, y, (ii) antisymmetric: xRy and yRx implies that $x = y$, and (iii) transitive: xRy and yRz implies xRz for any x, y, z.

According to SCRALA, the probability of an alternative x from a set of alternatives S and given a set of attributes A is the product of the probability of paying attention to the set of attributes in which x ranks first and the probability with which attention is not paid to other attributes in which x is not ranked first where the latter is weighted by $\frac{1}{|S|-1}$. Therefore, the choice probability is weighted down by the attributes in which it does not rank first and the number of alternatives in S. Note that $\gamma(a)$ is the probability that the relevant attribute a is considered or the attention probability of attribute a. The rule also considers high attention to the attributes, i.e., $\gamma(a) > \frac{1}{2}$, for all $a \in A$ where A is the relevant set of attributes. This assumption means that each *relevant* attribute is more likely to be considered than not. Another implication of this assumption is that if there is only one attribute then it selects the highest-ranked alternative in that attribute with a greater probability than the default alternative.

3 Axioms

We introduce the Axioms required to characterize SCRALA:

Axiom 1 (Independence (IN)) *An SCR p satisfies **IN** if for any $S \in \mathcal{X}$ and $C, D \in \mathcal{A}$ such that $C \cap D = \emptyset$, we have $p(x, S, C \cup D) = p(x, S, C)p(x, S, D)$.*

IN states that whenever two relevant attribute sets are disjoint, the choice probability for the union of the two sets is equal to the product of the choice probabilities given the two sets separately. The next Axiom applies to cases with a singleton attribute set.

Axiom 2 (Full Support (FS)) *An SCR p satisfies **FS** if for any $S \in \mathcal{X}$, $|S| \geq 2$ and any $a \in A$, $\sum_{x \in S} p(x, S, \{a\}) = 1$.*

FS states that under a single attribute and any set of alternatives, with at least two alternatives, the choice probabilities over the alternatives excluding the default alternative sum to one.

When there are two or more alternatives and a single attribute, the DM prefers flexibility in the menu of alternatives and the default alternative x^* is not chosen at all. However, if there is only one alternative and one attribute, then the DM may not choose that alternative with full support and the default alternative has a positive choice probability. This is due to the fact that there is no other alternative to compare with other than the default alternative which enhances the choice probability of the latter.

Axiom 3 (Invariance of Singletons (IS)) *An SCR p satisfies **IS** if for any $a \in A$ and $x, y \in X$, $p(x, \{x\}, \{a\}) = p(y, \{y\}, \{a\})$.*

IS states that if there is only one relevant attribute and only one alternative is available, then the choice probability of that alternative is independent of the alternative. In this case, the choice probability of the alternative is driven by the relevant attribute.

Axiom 4 (Uniformity in Inferior Alternatives (UIA)) *An SCR p satisfies* ***UIA*** *if for all $S \in \mathcal{X}$, $a \in A$,*

$$\big[\exists\, x, y, z \in S \ \ s.t.\ \ p(x, \{x, y\}, \{a\}) > p(y, \{x, y\}, \{a\})$$
$$and\ p(x, \{x, z\}, \{a\}) > p(z, \{x, z\}, \{a\})\big]$$
$$\implies \big[p(y, S, \{a\}) = p(z, S, \{a\})\big].$$

This Axiom states that (when a single attribute is considered) if there is an alternative x that has a higher choice probability than two other alternatives y, z when offered in two menus $\{x, y\}$ and $\{x, z\}$ respectively, then the choice probabilities of the latter two alternatives are equal in the menu S whenever $x, y, z \in S$. Since the choice probability of x is higher than those of y and z in the binary sets $\{x, y\}$ and $\{x, z\}$ respectively, y and z are inferior to x according to the choice probability in a binary comparison. This Axiom states that the inferior elements in a menu are uniform in terms of choice probability.

Axiom 5 (Dominance (DOM)) *An SCR p satisfies* ***DOM*** *if for all $a \in A$ and for all $S \in \mathcal{X}$ such that $x, y \in S$,*

$$\big[p(x, \{x, y\}, \{a\}) > p(y, \{x, y\}, \{a\})\ for\ all\ y \in S \setminus \{x\}\big]$$

$$\implies \big[p(x, S, \{a\}) = p(x, S \backslash \{y\}, \{a\})\ for\ all\ y \in S \setminus \{x\}\big].$$

DOM states that for any menu S and any alternative x when only one attribute is considered if the choice probability of x is strictly greater than that of y when offered in the menu $\{x, y\}$ for any $y \in S \backslash \{x\}$ then the choice probability of x does not change when y is removed from S with the same attribute being considered.

Axiom 6 (Binary difference (BD)) *An SCR p satisfies* ***BD*** *if for all $a \in A$ and for all $x, y \in X, x \neq y$, $p(x, \{x, y\}, \{a\}) \neq p(y, \{x, y\}, \{a\})$.*

This Axiom rules out equality of choice probabilities when the menu consists of two alternatives and a single attribute. In such situations, the DM chooses one of the two distinct alternatives with a strictly higher probability than the other.

Axiom 7 (Stochastic Transitivity (ST)) *An SCR p satisfies* ***ST*** *if for all $a \in A$*

and for all $x, y, z \in X$,

$$\Big[p(x, \{x, y\}, \{a\}) > p(y, \{x, y\}, \{a\}) \text{ and } p(y, \{y, z\}, \{a\}) > p(y, \{y, z\}, \{a\})\Big]$$

$$\implies p(x, \{x, z\}, \{a\}) > p(y, \{x, z\}, \{a\}).$$

This Axiom is a variation of Weak Stochastic Transitivity in Fishburn (1973) with strict inequalities and holds when a single attribute is considered. It is a straightforward version of transitivity modified for the attribute setting.

4 Results

We are now ready to state our main result.

Theorem 1 *An SCR satisfies* ***IN****,* ***FS****,* ***IS****,* ***UIA****,* ***DOM****,* ***BD*** *and* ***ST*** *if and only if it is a* ***SCRALA****.*

Proof. ($\implies$) Let p be an SCR that satisfies all the Axioms stated in the statement of Theorem 1. We define a binary relation on $\mathbf{X}$ for each attribute $a \in A$ and denote it as $\succeq_a$ as follows:

$$x \succeq_a y \iff p(x, \{x, y\}, \{a\}) \geq p(y, \{x, y\}, \{a\}).$$

We show that $\succeq_a$ is a well-defined linear ordering. We first show that $\succeq_a$ is complete. Take any $x, y \in \mathbf{X}$. If $x = y$, then $p(x, \{x, y\}, \{a\}) = p(y, \{x, y\}, \{a\})$. Therefore, $[x = y] \implies [x \succeq_a y$ and $y \succeq_a x]$. If $x \neq y$, by **BD**, $p(x, \{x, y\}, \{a\}) \neq p(y, \{x, y\}, \{a\})$. Therefore, either $p(x, \{x, y\}, \{a\}) > p((y, \{x, y\}, \{a\})$ which implies $x \succeq_a y$ or $p(y, \{x, y\}, \{a\}) > p(x, \{x, y\}, \{a\})$ which implies that $y \succeq_a x$. Therefore, $\succeq_a$ is complete.

We show that $\succeq_a$ is antisymmetric. Let $x \succeq_a y$ and $y \succeq_a x$ for some $x, y \in \mathbf{X}$. By definition of $\succeq_a$,

$$\big[x \succeq_a y\big] \implies [p(x, \{x, y\}, \{a\}) \geq p(y, \{x, y\}, \{a\})]$$

$$[y \succeq_a x] \implies \big[p(y, \{x, y\}, \{a\}) \geq p(x, \{x, y\}, \{a\})\big]$$

Above two statements imply that $p(x, \{x, y\}, \{a\}) = p(y, \{x, y\}, \{a\})$. However, by **BD** this can only be true if $x = y$. Hence, $\succeq_a$ is antisymmetric. The strict component of $\succeq_a$ can be derived as follows: $x \succ_a y \iff p(x, \{x, y\}, \{a\}) > p(y, \{x, y\}, \{a\})$ for any $x, y \in \mathbf{X}$ and $a \in A$.

For transitivity, let $x \succeq_a y$ and $y \succeq_a z$ for some $x, y, z \in \mathbf{X}$. Therefore, $p(x, \{x, y\}, \{a\}) \geq$

$p(y, \{x, y\}, \{a\})$ and $p(y, \{y, z\}, \{a\}) \geq p(z, \{y, z\}, \{a\})$. As x, y, z are distinct, by **BD**, the two aforementioned inequalities are strict. By **ST**,

$$\Big[p(x, \{x, y\}, \{a\}) > p(y, \{x, y\}, \{a\}) \text{ and } p(y, \{y, z\}, \{a\}) > p(z, \{y, z\}, \{a\})\Big]$$

$$\implies \big[p(x, \{x, z\}, \{a\}) > p(z, \{x, z\}, \{a\})\big].$$

Hence, $x \succeq_a z$ which establishes that $\succeq_a$ is transitive. Thus, $\succeq_a$ is a linear order.

Using the above binary relation $\succeq_a$, for any $S \in \mathcal{X}$, $A \in \mathcal{A}$, and $x \in S$, we define

$$A_x(S) := \{a \in A : x \succeq_a y, \forall y \in S\}$$

and

$$A_x^c(S) := \{a \in A : \exists y \in S \backslash \{x\} \text{ such that } y \succeq_a x\}.$$

Using **BD**, it can be verified that $A_x(S) \cap A_x^c(S) = \emptyset$. Since $A = A_x(S) \cup A_x^c(S)$, by **IN**, we have

$$p(x, S, A) = p(x, S, A_x(S))p(x, S, A_x^c(S)) \text{ for all } x \in S.$$

We now decompose the terms, $p(x, S, A_x(S))$ and $p(x, S, A_x^c(S))$ separately. By repeatedly applying **IN**,

$$p(x, S, A_x(S)) = \prod_{a \in A_x(S)} p(x, S, \{a\}) \text{ and } p(x, S, A_x^c(S)) = \prod_{b \in A_x^c(S)} p(x, S, \{b\}).$$

Therefore,

$$p(x, S, A) = \prod_{a \in A_x(S)} p(x, S, \{a\}) \prod_{b \in A_x^c(S)} p(x, S, \{b\}).$$

Let $S \backslash \{x\} = \{y_1, y_2, \cdots, y_{|S|-1}\}$. Suppose $a \in A_x(S)$. By definition of $A_x(S)$, $p(x, \{x, y_k\}, \{a\}) \geq p(y_k, \{x, y_k\}, \{a\})$ for all $k \in \{1, 2, \cdots, |S| - 1\}$. Moreover, since $x \notin \{y_1, \cdots, y_{|S|-1}\}$, by **BD**, $p(x, \{x, y_k\}, \{a\}) > p(y_k, \{x, y_k\}, \{a\})$ for all $k \in \{1, 2, \cdots, |S| - 1\}$. Therefore, $x \succ_a y_k$ for all $k \in \{1, 2, \cdots, |S| - 1\}$. By repeatedly applying **DOM**,

$$p(x, S, \{a\}) = p(x, S \backslash \{y_1\}, \{a\}) = p(x, S \backslash \{y_1, y_2\}, \{a\}) = \cdots = p(x, \{x\}, \{a\}). \quad (*)$$

By **IS**, for any $a \in A$, we can define $\gamma(a)$ as $\gamma(a) := p(w, \{w\}, \{a\})$ for any $w \in \mathbf{X}$.

Therefore, by **IN** and Equation $(*)$,

$$p(x, S, A_x(S)) = \prod_{a \in A_x(S)} p(x, \{x\}, \{a\}) = \prod_{a \in A_x(S)} \gamma(a).$$

Suppose $A_x^c(S) = \{a_1, a_2, \cdots, a_{|A_x^c(S)|}\}$. Let $y_1 = \max(\succeq_{a_1}, S)$ be the most preferred alternative in S according to $\succeq_{a_1}$, i.e., $y \succeq_{a_1} y'$ for all $y' \in S$. Similarly, let $y_k = \max(\succeq_{a_k}, S)$ for all $k \in \{1, 2, \cdots, |A_x^c(S)|\}$. Applying **FS**,

$$p(y_1, S, \{a_1\}) + \sum_{z \in S \setminus \{y_1\}} p(z, S, \{a_1\}) = 1$$

By **UIA**, for any $x, z \in S$, $[y_1 \succ_{a_1} x \text{ and } y_1 \succ_{a_1} z] \implies [p(x, S, \{a_1\}) = p(z, S, \{a_1\})]$. Therefore,

$$p(y_1, S, \{a_1\}) + (|S| - 1)p(x, S, \{a_1\}) = 1.$$

This implies that for $x \in S \setminus \{y_1\}$ such that $y_1 \succ_{a_1} x$, we have

$$p(x, S, \{a_1\}) = \frac{1 - p(y_1, S, \{a_1\})}{|S| - 1}.$$

Similarly, we can show that

$$p(x, S, \{a_k\}) = \frac{1 - p(y_k, S, \{a_k\})}{|S| - 1} \quad \text{for all } k \in \{1, 2, \cdots, |A_x^c(S)|\}.$$

Therefore,

$$p(x, S, \{a_k\}) = \frac{1 - \gamma(a_k)}{(|S| - 1)}, \text{ for all } a_k \in A_x^c(S). \quad (\dagger)$$

By **IN** and the above equation $(\dagger)$,

$$p(x, S, A_x^c(S)) = \frac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}.$$

Therefore,

$$p(x, S, A) = p(x, S, A_x(S))p(x, S, A_x^c(S)) = \prod_{a \in A_x(S)} \gamma(a) \frac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}.$$

If $A = A_x(S)$, by **IN**,

$$p(x, S, A) = p(x, S, A_x(S)) = \prod_{a \in A_x(S)} \gamma(a).$$

Similarly, if $A = A_x^c(S)$, by **IN**,

$$p(x, S, A) = p(x, S, A_x^c(S)) = \frac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}.$$

We now show that $\gamma(a) > \frac{1}{2}$ for all $a \in A$. By **BD**, for any $x, y \in \mathbf{X}$ with $x \neq y$, we have $p(x, \{x, y\}, \{a\}) \neq p(y, \{x, y\}, \{a\})$. W.l.o.g. suppose $p(x, \{x, y\}, \{a\}) > p(y, \{x, y\}, \{a\})$. By **FS**, $p(x, \{x, y\}, \{a\}) = 1 - p(y, \{x, y\}, \{a\}) > \frac{1}{2}$. Therefore, $\gamma(a) > \frac{1}{2}$ for all $a \in A$. Moreover, we have shown that $p = p^s$ is a SCRALA with $\succeq_a$ and $\gamma(a)$ as parameters for any $a \in A$. This completes the proof for the "Only If" part of the theorem.

($\Longleftarrow$) Now, we prove the "If" part for each Axiom separately. We show that any SCR, p^s which is a SCRALA satisfies all the Axioms mentioned in the statement of Theorem 1.

Claim 1 *SCRALA satisfies **IN**.*

Proof. Let p^s be a SCRALA. Suppose $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$. Assuming $A_x(S) \neq \emptyset$ and $B_x(S) \neq \emptyset$, for any $x \in S$, we have

$$p^s(x, S, A \cup B) = \prod_{a \in (A \cup B)_x(S)} \gamma(a) \frac{\prod_{b \in (A \cup B)_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|(A \cup B)_x^c(S)|}}$$

$$= \prod_{a \in A_x(S)} \gamma(a) \prod_{a \in B_x(S)} \gamma(a) \left(\frac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}} \right) \left(\frac{\prod_{b \in B_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|B_x^c(S)|}} \right)$$

$$= \left[\prod_{a \in A_x(S)} \gamma(a) \frac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}} \right] \left[\prod_{a \in B_x(S)} \gamma(a) \frac{\prod_{b \in B_x^c(S)} 1 - \gamma(b)}{(|S| - 1)^{B_x^c(S)|}} \right]$$

$$= p^s(x, S, A) p^s(x, S, B)$$

The other cases ($A_x(S) = A$ or $A_x^c(S) = A$) are much simpler and easy to verify.

Claim 2 *SCRALA satisfies **FS**.*

Proof. Let p^s be a SCRALA. Since S has finitely many elements, we can list S as $S = \{x_1, \cdots, x_{|S|}\}$. As $\succeq_a$ is a linear order over S, w.l.o.g we can order S as $x_1 \succeq_a x_2 \succeq_a \cdots \succeq_a x_{|S|}$. Then, $a \in A_{x_1}(S)$ and $a \in A_{x_i}^c(S), \forall i \in \{2, \cdots, |S|\}$. By the definition of SCRALA,

$$p^s(x_1, S, \{a\}) = \gamma(a)$$

and

$$p^s(x_i, S, \{a\}) = \frac{1-\gamma(a)}{|S|-1}, \text{ for all } i \in \{2, \cdots, |S|\}.$$

Therefore,

$$\begin{aligned}\sum_{x\in S} p^s(x, S, \{a\}) &= p^s(x_1, S, \{a\}) + \sum_{i=2}^{|S|} p^s(x_i, S, \{a\}) \\ &= \gamma(a) + \sum_{i=2}^{|S|} \frac{1-\gamma(a)}{|S|-1} \\ &= \gamma(a) + (|S|-1)\frac{1-\gamma(a)}{(|S|-1)} \\ &= \gamma(a) + 1 - \gamma(a) \\ &= 1\end{aligned}$$

Claim 3 *SCRALA satisfies **IS**.*

Proof. Let p^s be a SCRALA. Let $x, y \in \mathbf{X}$ and $a \in \mathbf{A}$. Letting $A = \{a\}$, we have $A_x(\{x\}) = A$ since $x \succeq_a x$ and $A_y(\{y\}) = A$ since $y \succeq_a y$. By the definition of SCRALA,

$$p^s(x, \{x\}, \{a\}) = \gamma(a) = p^s(y, \{y\}, \{a\}).$$

Claim 4 *SCRALA satisfies **UIA**.*

Proof. Consider any $S \in \mathcal{X}$ and $x, y, z \in S$ such that,

$$[p^s(x, \{x, y\}, \{a\}) > p^s(y, \{x, y\}, \{a\}) \text{ and } p^s(x, \{x, z\}, \{a\}) > p^s(z, \{x, z\}, \{a\}).$$

By the definition of SCRALA, $p^s(x, V, \{a\}) \in \{\gamma(a), 1-\gamma(a)\}$, where $V \in \mathcal{X}$ and $|V| = 2$. Since, $\gamma(a) > \frac{1}{2}, \forall a \in \mathbf{A}$, $p^s(x, \{x, y\}, \{a\}) = \gamma(a) = p^s(x, \{x, z\}, \{a\})$. Similarly, $p^s(y, \{x, y\}, \{a\}) = 1 - \gamma(a) = p^s(z, \{x, z\}, \{a\})$.

This implies that $x \succ_a y$ and $x \succ_a z$. Therefore, by definition of SCRALA,

$$p^s(y, S, \{a\}) = \frac{\gamma(a)}{|S|-1} = p^s(z, S, \{a\}).$$

Claim 5 *SCRALA satisfies **DOM**.*

Proof. Let p^s be a SCRALA. Consider any $a \in \mathbf{A}$, $S \in \mathcal{X}$ and $x \in S$ such that $p^s(x, \{x, y\}, \{a\}) > p^s(y, \{x, y\}, \{a\})$ for all $y \in S \setminus \{x\}$. This implies by SCRALA that $x \succ_a y$ for all $y \in S \setminus \{x\}$. Therefore,

$$p^s(x, S, \{a\}) = \gamma(a) = p^s(x, S \backslash \{y\}, \{a\}).$$

Claim 6 *SCRALA satisfies **BD**.*

Proof. Let p^s be a SCRALA. Consider any $a \in \mathbf{A}$ and $x, y \in \mathbf{X}$ with $x \neq y$. Since $\succeq_a$ is a linear order over X, either $x \succ_a y$ or $y \succ_a x$. W.l.o.g. suppose $x \succ_a y$. By the definition of SCRALA,

$$p^s(x, \{x, y\}, \{a\}) = \gamma(a) > \frac{1 - \gamma(a)}{2 - 1} = 1 - \gamma(a) = p^s(y, \{x, y\}, \{a\}).$$

Therefore, $p^s(x, \{x, y\}, \{a\}) \neq p^s(y, \{x, y\}, \{a\})$.

Claim 7 *SCRALA satisfies **ST**.*

Proof. Let p^s be a SCRALA. Suppose for some $x, y, z \in X$,

$$p^s(x, \{x, y\}, \{a\}) > p^s(y, \{x, y\}, \{a\}) \text{ and } p^s(y, \{y, z\}, \{a\}) > p^s(y, \{y, z\}, \{a\}).$$

Since p^s is a SCRALA with respect to $\succeq_a$ for any $a \in A$, the above two strict inequalities imply (as shown earlier) that $x \succ_a y$ and $y \succ_a z$. Since $\succ_a$ is transitive, $x \succ_a z$ which implies that $p(x, \{x, z\}, \{a\}) > p(z, \{x, z\}, \{a\})$. ■

4.1 Uniqueness Of Parameters

We now establish the uniqueness of parameters in the model which concern the salience of attributes and the (attributes) family of preferences over the alternatives, namely, $(\gamma(a))_{a \in \mathbf{A}}$ and $(\succeq_a)_{a \in \mathbf{A}}$.

Proposition 1 *The parameters $((\gamma(a))_{a \in \mathbf{A}}, (\succeq_a)_{a \in \mathbf{A}})$ for a SCRALA p^s are unique for any given choice data.*

Proof. Let p^s be a SCRALA that represents a choice data p. Suppose the family of parameters $((\gamma(a))_{a \in \mathbf{A}}, (\succeq_a)_{a \in \mathbf{A}})$ and $((\gamma^{'}(a))_{a \in \mathbf{A}}, (\succeq^{'}_a)_{a \in \mathbf{A}})$ both represent p under SCRALA p^s. We show that $((\gamma(a))_{a \in \mathbf{A}}, (\succeq_a)_{a \in \mathbf{A}})$= $((\gamma^{'}(a))_{a \in \mathbf{A}}, (\succeq^{'}_a)_{a \in \mathbf{A}})$. For any $x \in \mathbf{X}$, by definition of SCRALA, we have

$$p(x, \{x\}, \{a\}) = \gamma(a) = \gamma^{'}(a) = p(x, \{x\}, \{a\}), \forall\, a \in \mathbf{A}.$$

Thus, $(\gamma(a))_{a \in \mathbf{A}} = (\gamma^{'}(a))_{a \in \mathbf{A}}$.

Now, suppose that $(\succeq_a)_{a \in \mathbf{A}} \neq (\succeq^{'}_a)_{a \in \mathbf{A}}$. Then, $\exists\, x, y \in X$ with $x \neq y$ and $\exists\, a \in A$ such that $x \succeq_a y$ and $y \succeq^{'}_a x$. By SCRALA and by the fact that $\gamma(a)$ is unique, we have

$$p(x,\{x,y\},\{a\}) = \gamma(a) = \frac{1-\gamma(a)}{2-1} = 1-\gamma(a).$$

This implies that $\gamma(a) = \frac{1}{2}$. However, this is a contradiction to the definition of SCRALA since $\gamma(a) > \frac{1}{2}$ for all $a \in \mathbf{A}$. Hence, $(\succeq_a)_{a\in\mathbf{A}} = (\succeq'_a)_{a\in\mathbf{A}}$. ■

4.2 Regularity in Alternatives and in Attributes

We first show that SCRALA is regular in attributes.

Definition 3 (Regular in Attributes) *An SCR p is said to be Regular in Attributes if $p(x,S,A) \geq p(x,S,B)$, whenever $A,B \in \mathcal{A}$ with $A \subseteq B$, and $x \in S$.*

An interpretation of regularity in attributes is as follows. As more attributes become relevant, the choices become more difficult and the probability that an alternative is chosen weakly decreases (hence the probability that the empty set or the default alternative is chosen weakly increases) when the set of alternatives is fixed.

Proposition 2 *SCRALA satisfies Regularity in Attributes.*

Proof. Suppose p^s is a SCRALA and let $S \in \mathcal{X}$ and $A,B \in \mathcal{A}$ with $A \subseteq B$. By finiteness of B, we can write $B = A \cup \{b_1,\cdots,b_k\}$ for some $k \in \mathbf{N}$. Denote $B_1 = A \cup \{b_1\}$. Therefore,

$$p^s(x,S,B_1) = \begin{cases} \gamma(b_1)p^s(x,S,A) & \text{if } b_1 \in B_{1_x}(S); \\ \left(\frac{1-\gamma(b_1)}{|S|-1}\right) p^s(x,S,A) & \text{if } b_1 \in B^c_{1_x}(S). \end{cases}$$

Since $\gamma(b_1) \in (\frac{1}{2},1]$, we have $p^s(x,S,B_1) \leq p^s(x,S,A)$. Similarly, one can show that

$$p(x,S,B_k) \leq \cdots \leq p(x,S,B_2) \leq p(x,S,B_1) \leq p^s(x,S,B_0) = p^s(x,S,A),$$

where $B_0 = A$ and $B_i = B_{i-1} \cup \{b_i\}, \forall i \in \{1,\cdots,k\}$. Thus,

$$B_k = B_{k-1} \cup \{b_k\} = A \cup \{b_1,\cdots,b_{k-1}\} \cup \{b_k\} = B.$$

Consequently, $p^s(x,S,A) \geq p^s(x,S,B)$. This establishes the result. ■

Definition 4 (Regularity in Alternatives) *An SCR p satisfies Regularity in Alternatives if $p(x,S,A) \geq p(x,T,A)$, for all $A \in \mathcal{A}$ and for all $x \in S \subseteq T$, where $S,T \in \mathcal{X}$.*

This is the standard Regularity postulate which states that the choice probability of an alternative is weakly greater for a subset of a set when the set of attributes is fixed.

Proposition 3 *SCRALA satisfies Regularity in alternatives.*

Proof. Let $S, T \in \mathcal{X}$ with $S \subseteq T$. Assuming $A_x(S) \neq \emptyset$ and $A_x^c(S) \neq \emptyset$, for any $x \in S$, by SCRALA, we have

$$p^s(x, S, A) = \prod_{a \in A_x(S)} \gamma(a) \frac{\prod_{a \in A_x^c(S)}(1 - \gamma(a))}{(|S| - 1)^{|A_x^c(S)|}}$$

for any $A \in \mathcal{A}$. Since, T is finite, we can write $T = S \cup \{x_1, \cdots, x_k\}$ for some $k \in \mathbf{N}$. Denote $T_1 = S \cup \{x_1\}$. By the construction of the sets $A_x(\cdot)$ and $A_x^c(\cdot)$ any attribute $a \in A$ it is possible that $a \in A_x(S)$ and $a \in A_x^c(S \cup \{x_1\})$ (for instance if $\exists\ a \in A$ such that $x \succeq_a y, \forall y \in S$ but $x_1 \succeq_a x$). However, it is not possible that $a \in A_x^c(S)$ and $a \in A_x(S \cup \{x_1\})$ (even if $x \succeq_a x_1, \forall a \in A$). Combining the above with the fact that

$$\gamma(a) > \frac{1 - \gamma(a)}{|S| - 1} > \frac{1 - \gamma(a)}{|S|} = \frac{1 - \gamma(a)}{|S \cup \{x_1\}| - 1}, \forall a \in A,$$

(as $\gamma(a) > \frac{1}{2}$) we have $p^s(x, S, A) > p^s(x, S \cup \{x_1\}, A)$. Note that $p^s(x, S, A) = p^s(x, S \cup \{x_1\}, A)$ holds only when $x \succeq_a x_1, \forall a \in A$ and $A_x^c(S) = \emptyset$, the latter being necessary as

$$\frac{1 - \gamma(a)}{|S| - 1} > \frac{1 - \gamma(a)}{|S \cup \{x_1\}| - 1}.$$

Thus, $p^s(x, S, A) \geq p^s(x, S \cup \{x_1\}, A)$ Similarly, one can show that

$$p^s(x, T, A) = p^s(x, T_k, A) \leq \cdots \leq p^s(x, T_1, A) \leq p^s(x, T_0, A) = p^s(x, S, A),$$

where $T_0 = S$ and $T_i = T_{i-1} \cup \{x_i\}, \forall i \in \{1, \cdots, k\}$. Thus,

$$T_k = T_{k-1} \cup \{x_k\} = S \cup \{x_1, \cdots, x_{k-1}\} \cup \{x_k\} = T.$$

Therefore, $p^s(x, S, A) \geq p^s(x, T, A)$ whenever $S \subseteq T \in \mathcal{X}$. ■

SCRALA does not satisfy a stronger notion of regularity, which can be defined as follows.

Definition 5 (Regularity in Alternatives*) *An SCR p satisfies Regularity in Alternatives* if $p(x, S, \cdot) \geq p(x, S, \cdot)$ for all $x \in S \subseteq T \in \mathcal{X}$.*

Therefore, this version of regularity allows the attribute sets to be different. We show an example to show that SCRALA violates this stronger version of regularity in alternatives. Let $x \succeq_a z \succeq_a y$, $y \succeq_b x \succeq_b z$, $x \succeq_c y \succeq_c z$ and $\gamma(a) = 0.6, \gamma(b) = 0.7, \gamma(c) = 0.9$. Then by SCRALA, $p^s(x, \{x, y\}, \{a, b\}) = 0.18 < p^s(x, \{x, y, z\}, \{a, c\}) = 0.54$.

In the above example, we can make another observation that SCRALA can explain Asymmetric Dominance (or Attraction Effect) (see Huber et al. (1982)) which is described as follows: when the asymmetrically dominated element z is added to $\{x, y\}$, the choice probability of dominant element x is the highest in $\{x, y, z\}$ and in particular there is a reversal in choice probability ordering between x and y as the following calculations show:

$$p^s(x, \{x, y\}, \{a, b\}) = 0.18 < p^s(y, \{x, y\}, \{a, b\}) = 0.28$$

and

$$p^s(x, \{x, y, z\}, \{a, c\}) = 0.54 > p^s(y, \{x, y, z\}, \{a, c\}) = 0.01.$$

Note that in our example, it is crucial that the attributes sets are varied across contexts when the asymmetrically dominated alternative is added to the initial menu. One can interpret it as that when the asymmetrically dominated alternative (z) is added to the menu, $\{x, y\}$, it alters the relevant set of attributes (from $\{a, b\}$ to $\{a, c\}$) in favour of the asymmetrically dominant alternative (x).

4.3 Pareto Dominance and SCRALA

In this section, we establish that the deterministic variant of SCRALA satisfies Pareto Dominance. We first define Pareto Dominance for deterministic choice functions which is borrowed from the literature on choice functions (Arrow (1959) and Chernoff (1954)).

Let $c : \mathcal{X} \times \mathcal{A} \to \mathbf{X}$ be a deterministic choice function such that $C(S, A) \in S$ for any $S \in \mathcal{X}$ and $A \in \mathcal{A}$. Let $\succeq_a$ be the preference ordering according to an attribute $a \in A$. **Full attention** (FA): SCRALA satisfies FA if the DM pays full attention to all the attributes, i.e., $\gamma(a) = 1$ for all $a \in A$.

Definition 6 ***Pareto Dominance:*** *A deterministic choice function c satisfies Pareto dominance if for any $S \in \mathcal{X}$ and $A \in \mathcal{A}$ the following holds:*

(i) $[c(S, A) = \{x\}] \iff [x \succ_a y \;\; \forall\, a \in A, \forall\, y \in S \backslash \{x\}]$; *and*

(ii) $[c(S, A) = \{x^*\}] \iff [\forall\, x \in S \; \exists\, a \in A, y \in S \backslash \{x\}$ *such that* $y \succ_a x]$.

Pareto dominance requires that a choice is rationalizable by it if and only if the chosen alternative dominates *all* the other alternatives in the menu via *all* the relevant attributes. If there is no Pareto dominant element, then the choice set is empty. In terms of stochastic choice, the above translates into the fact that an alternative is chosen with probability 1 if and only if it dominates all other alternatives in the menu in all relevant attributes and given that $\gamma(a) = 1$ for all $a \in A$ (by FA). If there is no Pareto dominant element the default alternative is chosen with probability 1.

Let $c^s(S, A)$ denote the choice function derived from SCRALA as follows: (i) for any

$x \in X$, $c^s(S, A) = \{x\}$ if and only if $p^s(x, S, A) = 1$ (ii) otherwise $c^s(S, A) = \{x^*\}$. Note that by the definition of SCRALA and FA, $p(x, S, A) = 1$ if and only if $x \succ_a y$ for all $a \in A$ for all $y \in X \setminus \{x\}$. Consistent with the above, for any $a \in A$ we can define the attribute orderings $\succ_a$ over alternatives in revealed preference terms as: $x \succ_a y \iff \{x\} = c^s(\{x, y\}, \{a\}) \iff p(x, \{x, y\}, \{a\}) = 1, \forall x, y \in \mathbf{X}$.

Proposition 4 *Suppose SCRALA satisfies FA. The deterministic choice function c^s, based on SCRALA, satisfies Pareto Dominance.*

Proof. Take any $S \in \mathcal{X}$ and $A \in \mathcal{A}$. We first verify Condition (i) of Definition 6. To this end, suppose $x = c^s(S, A)$. By definition of SCRALA, it must be that $p^s(x, S, A) = 1$. However this is only possible if $x \succ_a y$ for all $a \in A$ and all $y \in S \backslash \{x\}$ and $\gamma(a) = 1$ for all $a \in A$. Therefore, $A_x(S) = A$. By definition of SCRALA,

$$p(x, S, A) = \prod_{a \in A} \gamma(a) = 1,$$

as $\gamma(a) = 1$ for all $a \in A$.

Similarly, one can show that

$$[x \succ_a y \; \forall \; a \in A \; \forall \; y \in S \backslash \{x\}] \implies [p^s(x, S, A) = 1] \implies [c^s(S, A) = x]$$

where the first implication holds under FA.

We now verify Condition (ii) of Definition 6. Suppose $c^s(S, A) = \{x^*\}$ for some $S \in \mathcal{X}$ and $A \in \mathcal{A}$. By definition of SCRALA, c^s and FA, we have $p^s(x, S, A) = 0$ for all $x \in S$. This implies that for each $x \in S$, $\exists \; a \in A$, $\exists \; y \in S \backslash \{x\}$ such that $y \succ_a x$. Conversely, assume that $\forall x \in S, \exists \; a \in A, \exists \; y \in S \backslash \{x\}$ such that $y \succ_a x$. This implies that $A^c_x(S) \neq \emptyset, \forall x \in S$. By SCRALA and FA,

$$\begin{aligned}
p^s(x, S, A) &= \prod_{a \in A_x(S)} \gamma(a) \frac{\prod_{a \in A^c_x(S)} (1 - \gamma(a))}{(|S| - 1)^{|A^c_x(S)|}} \\
&= \prod_{a \in A_x(S)} \gamma(a) \frac{\prod_{a \in A^c_x(S)} (1 - 1)}{(|S| - 1)^{|A^c_x(S)|}} \\
&= \prod_{a \in A_x(S)} \gamma(a) \prod_{a \in A^c_x(S)} 0 \\
&= 0.
\end{aligned}$$

By the definition of SCR,

$$\sum_{x \in S} p^s(x, S, A) + p^s(x^*, S, A) = 1.$$

Therefore, $p^s(x^*, S, A) = 1$ which implies that $c^s(S, A) = \{x^*\}$. ■

4.4 Application: A Social Choice Approach

We briefly touch on the heavily researched topic of Social Choice. Our spotlight, thus far, has been on the individual consumer and how she makes a choice from a set of alternatives. In the theory of social choice, we usually consider a society where there are many individuals voting to elect a candidate from a set of candidates. Each voter, in general, is assumed to have a preference over the contesting candidates. The preferences is then aggregated (sometimes into a social choice function/correspondence) in some suitable sense. The dominant strand of literature on Social Choice considers deterministic versions of the above aggregation. There is a strand of literature where the social choice functions are probabilistic. We seek to show that SCRALA can be applied to such scenarios.

Let $S \in \mathcal{X}$ denote the set of candidates, $A \in \mathcal{A}$ be the set of voters, and $\succeq_a$ be the preference ordering of a voter $a \in A$. In this setting, SCRALA can also be interpreted as a Probabilistic Social Choice Function, where the attributes are voters, and the orderings for attributes are viewed as orderings announced by the voters over the set of candidates. The parameters $(\gamma(a))_{a \in A}$ can be interpreted as the probability that voter a's preference is taken into account for making the social decision. Here $\gamma(a) \in (\frac{1}{2}, 1]$ implies that each voter's preference is more likely to be considered in the aggregation of preferences than not. We can restate SCRALA in this probabilistic social choice context as follows:

$$P^S(x, S, A) = \begin{cases} \prod_{a \in A_x(S)} \gamma(a) \dfrac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}, & \text{if } A_x(S) \neq \emptyset, A_x^c(S) \neq \emptyset; \\ \prod_{a \in A_x(S)} \gamma(a), & \text{if } A_x(S) = A; \\ \dfrac{\prod_{b \in A_x^c(S)} (1 - \gamma(b))}{(|S| - 1)^{|A_x^c(S)|}}, & \text{if } A_x^c(S) = A. \end{cases}$$

where

$$A_x(S) = \{a \in A : x \succeq_a y, \forall y \in S\}$$

denotes the voters who prefer candidate x over all the candidates in S and

$$A_x^c(S) = \{a \in A : \exists y \in S \backslash \{x\}, y \succeq_a x\}$$

is the set of voters where some candidate $y \in S \setminus \{x\}$ is preferred to candidate x where $x, y \in X$, $S \in \mathcal{X}$, $a \in A$, and $A \in \mathcal{A}$. Here, $P^S(x, S, A)$ denotes the probability that candidate x is the winner of the election when the menu of candidates are S and the set of voters are A. The probability that x^*(a default or status quo candidate or None Of The Above (NOTA)) is the outcome is: $P^S(x^*, S, A) = 1 - \sum_{x \in S} P^S(x, S, A)$.

Example 2 Suppose $X = \{x, y, z\}$ and $A = \{a_1, a_2\}$. Let the two orderings with respect to the agents a_1 and a_2 be $x \succ_{a_1} y \succ_{a_1} z$ and $x \succ_{a_2} z \succ_{a_2} y$ respectively. Let $\gamma(a_1) = 0.9$ and $\gamma(a_2) = 0.8$ be the two opinion consideration probabilities of the voters. The probabilities of being elected of the three candidates from the set $S = \{x, y, z\}$ according to SCRALA, P^S are as follows:

(i) $P^S(x, S, A) = \gamma(a_1)\gamma(a_2) = (0.9)(0.8) = 0.72.$

(ii) $P^S(y, S, A) = \left(\frac{1 - \gamma(a_1)}{|S| - 1}\right)\left(\frac{1 - \gamma(a_2)}{|S| - 1}\right) = \left(\frac{1 - 0.9}{3 - 1}\right)\left(\frac{1 - 0.8}{3 - 1}\right) = 0.005.$

(iii) $P^S(z, S, A) = \left(\frac{1 - \gamma(a_1)}{|S| - 1}\right)\left(\frac{1 - \gamma(a_2)}{|S| - 1}\right) = \left(\frac{1 - 0.9}{3 - 1}\right)\left(\frac{1 - 0.8}{3 - 1}\right) = 0.005.$

(iv) $P^S(x^*, S, A) = 1 - \sum_{x \in S} p(x, S, A) = 1 - 0.72 - 0.005 - 0.005 = 0.27$

Note that the probability of candidate x being elected is the highest by a large magnitude (in comparison with y and z) as it is the most preferred alternative for both voter a_1 and a_2. But she is not elected with probability 1 because there is a non-zero probability that voter a_1's opinion is ignored ($= 1 - \gamma(a_1) = 0.1$) and there is a non-zero probability that voter a_2's opinion is ignored ($= 1 - \gamma(a_2) = 0.2$). Hence, the default (status quo or NOTA) is the outcome with a probability of 0.27.

5 Conclusion

We provide a model of individual stochastic choice where a decision maker is attention-biased towards different attributes where the attributes are considered with independent probabilities. For future work, it would be interesting to explore a framework where the attention probabilities are dependent. For example, it is possible that if attention paid to an attribute is high then the attention paid to another complementary attribute is also high. This can happen when two or more attributes are related, for example, while buying a box of breakfast cereal, the 'type of ingredients' as an attribute may be linked to another attribute, 'the sugar content' of the cereal.

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6 Appendix

6.1 Independence of the Axioms

In this section, we prove the independence of the Axioms. To show this, we provide an example of an SCR for each Axiom listed in the statement of the Theorem 1 which satisfies all Axioms except that one.

Claim 8 *The following SCR-1 satisfies all the Axioms except **IN**.*

Definition 7 (SCR-1) *An SCR, p^1 is a SCR-1 if for all $a \in \mathbf{A}$ there exist an ordering $\succeq_a$, an attention parameter $\gamma(a) \in (\frac{1}{2}, 1]$ and an ordering $\succcurlyeq$ on $\mathbf{A}$ such that for any $S \in \mathcal{X}$, $x \in S$ and $A \in \mathcal{A}$,*

$$p^1(x, S, A) = \begin{cases} \gamma(a^*) & \text{if } a^* \in A_x(S) \text{ where } a^* = \max(\succcurlyeq, A); \\ \frac{1-\gamma(a^*)}{|S|-1} & \text{otherwise.} \end{cases}$$

For an example, consider $A = \{a_1, a_2\}$ with $x \succeq_{a_1} y \succeq_{a_1} z$, $y \succeq_{a_2} x \succeq_{a_2} z$, and $a_1 \succcurlyeq a_2$, then for $S = \{x, y, z\}$,

$$p^1(x, S, A) = \gamma(a_1),\ p^1(y, S, A) = p^1(z, S, A) = \frac{1 - \gamma(a_1)}{2}$$

while,

$$p^1(x, \{x, y\}, \{a_1\}) = p^1(x, \{x\}, \{a_1\}) = \gamma(a_1)$$

and

$$p^1(x, \{x, y\}, \{a_2\}) = 1 - \gamma(a_2),\ p^1(y, \{y\}, \{a_2\}) = p(x, \{x\}, a_2) = \gamma(a_2).$$

It is easy to verify that the rule does not satisfy IN since

$$p^1(x, S, A) = \gamma(a_1) \neq p^1(x, S, \{a_1\})p^1(x, S, \{a_2\}) = \gamma(a_1)\left(\frac{1 - \gamma(a_2)}{|S| - 1}\right).$$

The rule satisfies all the other Axioms.

Claim 9 *The following SCR-2 satisfies all the Axioms except **FS**.*

Definition 8 (SCR-2) *An SCR, p^2 is SCR-2 if for all $a \in \mathbf{A}$ there exists an ordering $\succeq_a$ and an attention parameters $\gamma(a) \in (\frac{1}{2}, 1]$ such that for any $S \in \mathcal{X}$,*

$x \in S$ and $A \in \mathcal{A}$,

$$p^2(x,S,A) = \begin{cases} \prod_{a\in A_x(S)}\gamma(a)\dfrac{\prod_{b\in A_x^c(S)}(1-\gamma(b))}{2(|S|-1)^{|A_x^c(S)|}}, & \text{if } A_x(S)\neq\emptyset, A_x^c(S)\neq\emptyset; \\ \prod_{a\in A_x(S)}\gamma(a), & \text{if } A_x(S)=A; \\ \dfrac{\prod_{b\in A_x^c(S)}(1-\gamma(b))}{2(|S|-1)^{|A_x^c(S)|}}, & \text{if } A_x^c(S)=A. \end{cases}$$

Consider $S = \{x, y\}$ and $A = \{a\}$ such that $x \succ_a y$. This rule does not satisfy **FS** since

$$p^2(x,\{x,y\},\{a\}) = \gamma(a),\ p^2(y,\{x,y\},\{a\}) = \frac{1-\gamma(a)}{2},\ \text{and } p^2(x^*,\{x,y\},\{a\}) = \frac{1-\gamma(a)}{2}.$$

It is easy to verify that SCR-2 satisfies all the other Axioms.

Claim 10 *The following SCR-3 satisfies all the Axioms except **IS**.*

Definition 9 (SCR-3) *An SCR, p^3 is SCR-3 if there exists an ordering $\succeq_a$ and attention parameters $\gamma(a) \in (\frac{1}{2}, 1]$ for all $a \in \mathbf{A}$ such that for any $S \in \mathcal{X}$, $x \in S$, and $A \in \mathcal{A}$,*

$$p^3(x,S,A) = \begin{cases} p^s(x,S,A), & \text{if } |S|\geq 2; \\ \prod_{a\in A_x(\mathbf{X})}\gamma(a), & \text{if } A_x(\mathbf{X})=A, |S|=1; \\ \prod_{a\in A_x^c(\mathbf{X})}(1-\gamma(a)), & \text{if } A_x^c(\mathbf{X})=A, |S|=1; \\ \prod_{a\in A_x(\mathbf{X})}\gamma(a)\prod_{b\in A_x^c(\mathbf{X})}(1-\gamma(b)), & \text{if } A_x(\mathbf{X})\neq\emptyset, A_x^c(\mathbf{X})\neq\emptyset, |S|=1. \end{cases}$$

Consider $\mathbf{X} = \{x, y\}$ and $A = \{a\}$ such that $x \succ_a y$. This rule does not satisfy **IS** since

$$p^3(x,\{x\},\{a\}) = \gamma(a),\ p^3(y,\{y\},\{a\}) = 1-\gamma(a).$$

It is easy to verify that SCR-3 satisfies all the other Axioms.

Claim 11 *The following SCR-4 satisfies all the Axioms except **UIA**.*

Let

$$A_x^1(S) = \{a \in A : x = \max(\succeq_a, S)\}$$

and

$$A_x^2(S) = \{a \in A : x = \max(\succeq_a, S\backslash\{x_a^*(S)\})\}.$$

where $x_a^*(S) = \max(\succeq_a, S)$.

Definition 10 (SCR-4) *An SCR, p^4 is SCR-4 if for all $a \in \mathbf{A}$ there exists an ordering $\succeq_a$ and attention parameters $\gamma(a) \in (\frac{1}{2}, 1]$ such that for any $S \in \mathcal{X}$, $x \in S$*

and $A \in \mathcal{A}$,

$$p^4(x,S,A) = \begin{cases} \prod_{a\in A^1_x(S)} \gamma(a)\dfrac{\prod_{b\in A^2_x(S)}(1-\gamma(b))}{(|S|-1)^{|A^2_x(S)|}}, & \textit{if } A^1_x(S)\neq\emptyset, A^2_x(S)\neq\emptyset;\\ \prod_{a\in A^1_x(S)} \gamma(a), & \textit{if } A^1_x(S)=A;\\ \dfrac{\prod_{b\in A^2_x(S)}(1-\gamma(b))}{(|S|-1)^{|A^2_x(S)|}}, & \textit{if } A^2_x(S)=A;\\ 0, & \textit{otherwise.}\end{cases}$$

Consider $S = \{x, y, z\}$ and $A = \{a\}$ such that $x \succ_a y \succ_a z$. This rule does not satisfy **UIA** since

$$p^4(x,S,A) = \gamma(a),\ p^4(y,S,A) = 1-\gamma(a) \text{ and } p(z,S,A) = 0.$$

It is easy to verify that SCR-4 satisfies all the other Axioms.

Claim 12 *The following SCR-5 satisfies all the Axioms except **DOM**.*

Definition 11 (SCR-5) *An SCR p^5 is SCR-5 if for all $a \in \mathbf{A}$ there exists an ordering $\succeq_a$ and attention parameters $\gamma(a) \in (\frac{1}{2}, 1]$ such that for any $S \in \mathcal{X}$, $x \in S$ and $A \in \mathcal{A}$,*

$$p^5(x,S,A) = \begin{cases} p^s(x,S,A), & \textit{if } |S|\geq 2;\\ 1, & \textit{if } |S|=1.\end{cases}$$

SCR-5 is a SCRALA if $|S| \geq 2$ and chooses an alternative with probability one from any singleton set.

Consider $S = \{x, y, z\}$ and $A = \{a\}$ such that $x \succ_a y \succ_a z$. This rule does not satisfy **DOM** since

$$p^5(x,\{x,y\},\{a\}) = \gamma(a) \neq p^5(x,\{x\},\{a\}) = 1.$$

It is easy to verify that SCR-5 satisfies all the other Axioms.

Claim 13 *The following SCR-6 satisfies all the Axioms except **BD**.*

Definition 12 (SCR-6) *An SCR p^6 is SCR-6 if for all $a \in \mathbf{A}$ there exists an ordering $\succeq_a$ and attention parameters $\gamma(a) \in (\frac{1}{2}, 1]$ such that for any $S \in \mathcal{X}$, $x \in S$ and*

$A \in \mathcal{A}$,

$$p^6(x,S,A) = \begin{cases} \prod_{a\in A_x(S)} \gamma(a) \dfrac{\prod_{b\in A_x^c(S)}(1-\gamma(b))}{(|S|-1)^{|A_x^c(S)|}}, & \textit{if } A_x(S) \neq \emptyset, A_x^c(S) \neq \emptyset; \\ \prod_{a\in A_x(S)} \gamma(a), & \textit{if } A_x(S) = A; \\ \dfrac{\prod_{b\in A_x^c(S)}(1-\gamma(b))}{(|S|-1)^{|A_x^c(S)|}}, & \textit{if } A_x^c(S) = A. \end{cases}$$

SCR-6 is a SCRALA with $\gamma(a) = \frac{1}{2}$ for all $a \in A$.

Consider $S = \{x, y\}$ and $A = \{a\}$ such that $x \succ_a y$. This rule does not satisfy **BD** since

$$p^6(x, \{x,y\}, \{a\}) = p^6(y, \{x,y\}, \{a\}) = \frac{1}{2}.$$

It is easy to verify that SCR-6 satisfies all the other Axioms.

Claim 14 *The following example of an SCR satisfies all the Axioms except* ***ST*** *for* $S = \{x, y, z\}$ *and* $A = \{a\}$.

Proof. Consider $X = \{x, y, z\}$ and $A = \{a\}$. Let the SCR be as follows:

$$p(x, X, A) = p(z, X, A) = \frac{1}{3},\ p(y, X, A) = \frac{1}{3}.$$

$$p(x, \{x,y\}, A) = \frac{2}{3},\ p(y, \{x,y,\}, A) = \frac{1}{3}.$$

$$p(y, \{y,z\}, A) = \frac{2}{3},\ p(z, \{y,z\}, A) = \frac{1}{3}.$$

$$p(z, \{x,z\}, A) = \frac{2}{3},\ p(x, \{x,z\}, A) = \frac{1}{3}.$$

$$p(x, \{x\}, A) = p(y, \{y\}, A) = p(z, \{z\}, A) = \frac{2}{3}.$$

Note that **UIA**, **DOM** and **IN** are satisfied trivially, while **FS**, **IS** and **BD** are satisfied by the definition of the SCR above. ■