Ashoka University Economics
Discussion Paper 96

Contraction consistent aggregation on trees

August 2023

Mihir Bhattacharya, Ashoka University

Contraction consistent aggregation on trees

Mihir Bhattacharya *

August 9, 2023

Abstract

We study *contraction consistent* social choice functions (s.c.f.) in a setting where voters have single-peaked preferences over a tree. This is relevant in settings where alternatives are locations spread out on a tree and a location needs to be selected for provision of a public good. An s.c.f. is *contraction consistent* if its outcome at any profile does not change when the profile is restricted to any subset consisting of the outcome. We show that *q-threshold rules on trees* are the only s.c.f.s which satisfy *contraction consistency*, *unanimity* and *anonymity*. These s.c.f.s specify, for each alternative, thresholds which are decreasing (increasing) on every path from a given node. These s.c.f.s then select from the range, the unique alternative which is the smallest (greatest) alternative in any restricted vote profile that receives more additive votes than the threshold assigned to it. These s.c.f.s are generalizations of *min*, *max*, and *median* s.c.f.s when restricted to paths.

JEL classification: D70, D71

Keywords: contraction consistency, social choice functions, single-peaked preferences, trees.

*Department of Economics, Ashoka University, Rajiv Gandhi Education City, Rai, Sonipat, Haryana-131029, INDIA. Email: mihir.bhattacharya@ashoka.edu.in
Conflicts of interest: None.

1 Introduction

In many voting situations, candidates or alternatives may drop out or become unavailable due to certain unforeseen circumstances. In such cases, the aggregation rule must specify what the outcome would be for every such contingency. In this paper we focus on a consistency condition which is crucial under such conditions. *Contraction consistency* states that the outcome of a social choice function (s.c.f.) at any preference profile should be the same when *restricted* to any subset containing that outcome. This can also be interpreted as the social choice variant of Sen (1969) and Chernoff (1954)'s *contraction consistency* or α condition defined for individual choice functions. In this paper we study *contraction consistent* social choice functions when the preferences are single-peaked over a tree.[1]

Alternatives are often distributed over a tree in many settings. For example,

(1) When alternatives are locations in a city with many 'branches' and a public good needs to be provided at one of these locations.

(2) Alternatives may be researchers or professionals connected by relevance (like a tree) and one of them is to be selected for an award.

Single-peakedness is a natural assumption to make in these settings. It requires that individuals have a 'peak' or a most preferred policy and that alternatives closer to the peak are strictly preferred over the ones further away. In the first example, the individuals would always prefer to have the allocated good at a location closest to their own location and in the second example, individuals would prefer that an individual closest to their area of interest be awarded. Single-peaked preferences on a line were first introduced by Black et al. (1958), Hotelling (1929) and Downs (1957) and continue to be used in political economy as well as social choice settings.[2] Single-peakedness on a tree was first introduced by Demange (1982), and this notion of single-peakedness was generalized by Nehring and Puppe (2007) and Nehring and Puppe (2010).

There are a few works which study this type of consistency of social choice functions in the unrestricted domain. Blair et al. (1976) provide different impossibility theorems using *Chernoff's conditions aka contraction consistency* along with other axioms like *independence of irrelevant alternatives* and *path independence*. Bandyopadhyay (1984) finds that instead of contraction consistency it is expansion consistency which is problematic for the existence of non-dictatorial social choice rules. We provide an

[1] A formal definition is provided in the model section. Intuitively, a preference is single-peaked over a tree, if (i) there exists a peak of the preference x which is strictly preferred over every other alternative and (ii) alternatives further away from the peak in any 'direction' are strictly worse-off.

[2] See Arrow et al. (2010) and Austen-Smith and Banks (2005) for further exposition on social choice models with single-peaked preferences.

example to illustrate our main axiom.

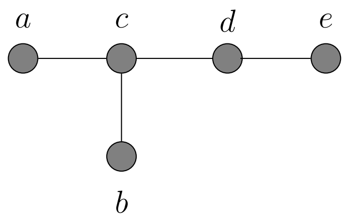

Figure 1: Single-peaked preference on T

Example 1 *Suppose the set of alternatives is* $X = \{a, b, c, d, e\}$ *with tree* T *as illustrated in Figure 1 and let the set of voters be* $N = \{1, 2, 3\}$. *Suppose the voter preferences are as follows:*

$\succ_1$	$\succ_2$	$\succ_3$
a	c	d
c	a	e
b	d	c
d	b	b
e	e	a

Note that preferences are single-peaked preferences on the given tree since alternatives further 'away' from the tree are strictly worse-off. Let this profile of preferences be denoted by π *and suppose the outcome is* $f(\pi) = d$. *Contraction consistency requires that restricting the profile to any set of alternatives* S *which contain* d *should not change the outcome. For example,* $f(\pi_{\{a,b,d\}}) = f(\pi_{\{a,b,c,d\}}) = d$ *and so on. Suppose the aggregation rule only cares about the top-ranked alternatives (we will show that contraction consistent and unanimous rules will satisfy this property). In that case, we can write only the top-ranked alternatives as inputs for aggregation, i.e.,* $f(\pi) = f(a, c, d) = d$. *By contraction consistency,* $f(\pi_{\{a,b,d\}}) = f(a, a, d) = d$. *Note that since* c *is no longer available, voter* 2*'s top ranked alternative in the restricted profile to the set* $\{a, b, d\}$ *is* a. *Further observations can be made on similar restrictions of the profile.*

An axiom that is similar to ours is *self-selectivity* as introduced by Koray (2000) and further extended to single-peaked domains by Bhattacharya (2019).[3] A social choice function is *self-selective* if it chooses itself when subjected to a 'vote' against other voting rules. Both the notions are 'consequentialist' in the sense that they only care about the alternative being chosen. Therefore, when the set of available voting rules changes, it is equivalent to a contraction of the set of alternatives over which the profile

[3]Koray and Unel (2003) extends the analysis to the *tops-only* domain and Koray and Slinko (2008) characterizes self-selective rules when the inefficient ones are excluded.

is constructed. Lainé et al. (2016) study a similar notion of stability for scoring rules and social welfare functions, and show that there are no *hyper-stable* scoring rules. Barberà and Jackson (2004) study a version of self-stability of voting rules over two alternatives, and find that majority rules are the only self-stable rules. Our notion of consistency is closer to the literature on self-selectivity than the earlier literature on contraction consistency since the nature of preferences plays an important role in the characterization results.

We characterize *q-threshold rules on trees* which are defined as follows. Pick any terminal node or alternative r on the tree. Assign monotone decreasing thresholds to each alternative on every path $[r, y]$ for any other terminal node y. These rules for every preference profile π pick the unique alternative $x^*(\pi)$ which when restricted to any path consisting of $x^*(\pi)$ is such that it is the smallest alternative from the *range* of the profile that receives more cumulative votes at the top (votes received by all the alternatives which are closer to r) than the threshold assigned to them.[4] The thresholds are defined for each alternative and depend on the path they are defined over.

Another condition that needs to be satisfied for the above rule to be consistent with itself is as follows. Suppose x is an alternative which has a degree greater than or equal to 3, and $q_x^{[r,y']}$ and $q_x^{[r,y'']}$ are the two thresholds of x in two extremal paths $[r, y']$ and $[r, y'']$. The condition requires that $q_x^{[r,y']} + q_x^{[r,y'']} < n + 2$. This condition ensures that the rule satisfies contraction consistency.

q-threshold rules can be seen as generalized *quota rules* defined for trees but with variable thresholds which apply to cumulative votes for alternatives at the top. These rules can be defined with respect to any terminal node r and by assigning decreasing or increasing thresholds to alternatives on every extremal path from r.[5]

We prove our main result in steps. We first show that if a social choice function is *contraction consistent* and *unanimous* then it is *tops-only* (Proposition 1). *Contraction consistency* then implies that any such rule which is *unanimous* will only choose from the range of the profile. Finally, we use the fact that the two versions of *contraction consistency* are the same under the tops-only property. The main proof overcomes significant challenges since the rules must be consistent across different extremal paths in the tree. Once that is resolved, we show that *q-threshold rules over trees* are the only *contraction consistent*, *anonymous* and *unanimous* social choice functions over trees.

[4] The *range* of the profile is the set containing alternatives which lie on the path between at least one pair of alternatives at the top of the profile. Alternatively these rules can be defined with respect to monotone increasing thresholds and adding the cumulative votes of alternative further away from r.

[5] A path $[a, b]$ is *extremal* if both a and b are terminal nodes.

There are some differences between our notion of consistency and the ones in the literature. First, our definition of *contraction consistency* is applied on restriction of preference profiles, while the literature mostly defines self-selection using an induced profile over different aggregation rules and then use neutrality to check for the self-selectivity. Second, the standard notion of contraction consistency applied in the classical social choice literature (Bandyopadhyay (1984), Blair et al. (1976)) apply the condition to the social decision function without describing the preferences of individuals over the contracted set. In our notion, we implicitly assume that individuals are truthful and do not change their preferences over the feasible set of alternatives. Moreover, the structure of preferences is crucial for obtaining the characterization of q-threshold rules.

An important contribution of the paper is that it provides a characterization of a large class of social choice functions in a relatively general domain which has shown promise for existence results in the literature. The fact that the domain is partially ordered helps in obtaining such results. Another important insight of our paper is that rules which are contraction consistent and unanimous are also tops-only.

The paper is organised as follows. In Section 2 we introduce the notation and describe the model. Section 3 lists the Axioms and Section 4 provides the results. This is followed by some concluding remarks. The proofs of all the results are provided in the Appendix. The bibliography is provided at the end.

2 The Model

In this section we describe the model and provide all the definitions. There is a finite set of voters $N = \{1, 2, \ldots, n\}$ and a finite set of alternatives X with $|X| = m$. Let $\mathcal{P}(X)$ denote the set of all non-empty subsets of X. We assume that all the alternatives are placed on an undirected *tree*, $T \equiv T(X, E)$, where the set of nodes is the set of alternatives X and E is the set of edges. We will assume the tree $T(X, E)$ to be fixed for the remaining part of the paper. We will use the terms 'alternatives' and 'nodes' interchangeably.

An *path* $[x, y]$ from node x to node y in X, $x \neq y$ is a sequence of distinct nodes $(x_0, x_1, ..., x_k)$ such that $x_0 = x$, $x_k = y$ and $\{x_q, x_{q+1}\} \in E$ for all $q \in \{0, 1, ..., k-1\}$. For simplicity of notation, we also denote $[x, y]$ as the set of alternatives (including x and y) in the path $[x, y]$ for any distinct $x, y \in X$. The *degree* of a node is the number of other nodes it is connected to. Let $deg(x)$ denote the degree of the alternative (or node) $x \in X$ in T. A node $x \in X$ is an end node if $deg(x) = 1$. A path $[a, b]$ is *extremal* if both a and b are end nodes. Let $\mathbf{E}$ denote the set of all extremal paths in T. An alternative x is *adjacent* to x^+ if $[x, x^+] = \{x, x^+\}$ i.e. there are no other distinct alternatives on the path $[x, x^+]$ other than x and x^+. We now define single-peakedness over a tree based on Demange (1982).

Single-peaked preferences over trees: Each voter $i \in N$ has a single-peaked preference on T which is defined as follows. A strict preference ordering, $\succ_i$, is *single-peaked on the tree* T if there exists a 'peak' $x_i \in X$ such that for all $x, y \in X, x \neq y$,[6]

$$x \in [x_i, y] \Rightarrow x \succ_i y.$$

Note that in the above definition if $x = x_i$ then for any $y \neq x_i$ we have $x_i \succ_i y$. This definition reduces to the standard definition of single-peaked preferences if the tree T is a 'line'.

Therefore, alternatives closer to the peaks are strictly preferred over the alternatives further away. However, the definition does not impose any restriction on two alternatives which are on either side of the peak or, in other words, when neither is in the path between the peak and the other alternative. For example, in Figure 1 voter i with peak a can have either $b \succ_i d$ or $d \succ_i b$.

We consider the whole set of strict single-peaked preference orderings on trees. For example, suppose $X = \{a, b, c, d, e\}$ and the tree is T as shown in Figure 1. Then, the set of all strict single-peaked preferences on the given tree with peak a include the strict preferences *acbde*, *acdbe*, and *acdeb* where alternatives are in the decreasing order of strict preference.[7]

Let $\mathcal{S}(T)$ be the set of all single-peaked preference orderings on T. A *preference profile* $\pi = (\succ_1, \ldots, \succ_n) \in \mathcal{S}(T)^n$ is a collection of n preferences with peaks or 'tops', $\tau(\pi) = (\tau_1(\pi), ..., \tau_n(\pi))$. Let $\tau_i(\pi)$ denote the top of voter i in the profile π. We denote by π_S as the restriction of the profile π to a subset $S \in \mathcal{P}(X)$. For any $S \in \mathcal{P}(X)$ let $\mathcal{S}(T_S)^n$ be the set of all profiles which are restrictions of profiles in $\mathcal{S}(T)^n$ to the set S. We will write $\mathcal{S}(T)$ instead of $\mathcal{S}(T_X)$ for simplicity.

Social choice function (s.c.f.): A *social choice function (s.c.f.)* is a mapping $f : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ such that for any profile $\pi_S \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$ we have $f(\pi_S) \in S$. Therefore, an s.c.f. operates on every restriction of a single-peaked profile on T to any subset S of X and produces an alternative in S.

[6] A strict binary relation $\succ$ is an ordering if it is: (i) Complete: For all $x, y \in X$, $x \neq y$ either $x \succ y$ or $y \succ x$ (ii) Transitive: If for all $x, y, z \in X$ $x \succ y$ and $y \succ z$ implies $x \succ z$, and (iii) Irreflexive: $\neg[x \succ x]$ for all $x \in X$.

[7] Single-peaked preferences on trees defined in Schummer and Vohra (2002) are based on a notion of 'distance' and, therefore, the preferences in their model are uniquely identified by the 'peaks' of the individuals.

In this paper, we will only focus on *tops-only* s.c.f.s since our main axiom, *contraction consistency*, along with *unanimity* will imply this property. We define the *tops-only* property below.

Tops-only. An s.c.f. f satisfies *tops-only* if for all $\pi, \pi' \in \mathcal{S}(T_S)^n$ such that $\tau(\pi) = \tau(\pi')$,

$$f(\pi) = f(\pi').$$

For simplicity, we will write $f(\tau(\pi))$ in place of $f(\pi)$ for tops-only rules. This is a useful property in our model since to arrive at the restriction of a profile $\pi \in \mathcal{S}(T)^n$ to a subset $S \in \mathcal{P}(X)$ of alternatives, we only need to consider the changes in the top-ranked alternatives of all the voters. For example, for $X = \{a, b, c, d, e\}$ and 5 voters with the given tree T as in Fig. 1, if π is such that $\tau(\pi) = (a, b, c, e, e)$, then $\tau(\pi_{[a,e]}) = (a, c, c, e, e)$. To see this, note that voter 2's peak from the full set of alternatives is b so by single-peakedness over T, she will prefer c to all the other alternatives in $[a, e]$. Therefore, her top-ranked alternative from $[a, b]$ is c. Every other voter's top-ranked alternative is available in $[a, e]$ so $\tau(\pi_{[a,b]}) = (a, c, c, e, e)$. We will use this property throughout the rest of the paper to compute restrictions of a given profile to different subsets including paths in the tree T. We provide some examples of such s.c.f.s in this setting.

Dictatorial s.c.f. An s.c.f. $f^i : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$, for a given voter $i \in N$ is dictatorial if $f^i(\pi) = \tau_i(\pi)$ for all $\pi \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$.

For example, suppose $X = \{a, b, c, d, e\}$ and $N = \{1, 2, ..., 5\}$ with the tree T as given in Fig. 1. Then, for the dictatorial rule f^1, we have $f^1(\tau_1(\pi), \tau_2(\pi), ..., \tau_5(\pi)) = \tau_1(\pi)$ for all $\pi \in \mathcal{S}(T_S)^n$. For example, if $\tau(\pi) = (a, b, c, e, e)$, then $f^1(\pi) = a$.

Positional rules: An s.c.f. $f^k_{[r,y]} : \bigcup_{S \in \mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$, for a given $k \in \mathbb{R}$ for a given $[r, y] \in \mathbf{E}$ is a positional rule if $f^k_{[r,y]}(\pi) = \tau^*_k(\pi_{[r,y]})$ for all $\pi \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$ where τ^*_k is the alternative in the $k-$th voter's top-ranked alternative when arranged from r to y for $(k \in \{1, 2, ..., n\})$ when the profile π is restricted to the path $[r, y]$. Min, max and median rules or a mixture of these rules can be defined using any path $[r, y] \in \mathbf{E}$.

For example, suppose $X = \{a, b, c, d, e\}$ and $N = \{1, 2, ..., 5\}$ with the tree T as given in Fig. 1. Then, for the positional rule $f^2_{[a,e]}$, we have $f^2_{[a,b]}(\tau_1(\pi), \tau_2(\pi), ..., \tau_5(\pi)) = \tau^*_2(\pi_{[a,b]})$ for all $\pi \in \mathcal{S}(T_S)^n$. For example, if $\tau(\pi) = (a, b, c, e, e)$, then $\tau(\pi_{[a,b]}) =$

(a, b, c, b, b) and $f^2_{[a,b]}(\pi) = b$.

We introduce some definitions for our next rule which is a generalized version of these rules.

Range of a profile: For any profile $\pi_S \in \mathcal{S}(T_S)^n$, the *range* of the profile is the set of all alternatives that lie on the path between a pair of top-ranked alternatives in $\tau(\pi_S)$, i.e.,

$$Range(\pi_S) = \{x \in S : x \in [\tau_i(\pi_S), \tau_j(\pi_S)] \text{ for some } i, j \in N\}.$$

For simplicity, we denote $Range(\pi)$ as the range of the full profile $\pi \in \mathcal{S}(T)^n$. In Figure 1, for a three-voter profile $\pi \in \mathcal{S}(T)^3$ with $\tau(\pi) = (a, b, d)$, $Range(\pi) = \{a, b, c, d\}$. We introduce some notation to define our next rule.

Thresholds on $[r, y]$: For an extremal path $[r, y] \in \mathbf{E}$, we define thresholds $q^{[r,y]} : X \to N$ which will be *monotone decreasing* on $[r, y]$, i.e., $[x \in [r, x']] \Rightarrow [q^{[r,y]}_x \geq q^{[r,y]}_{x'}]$ for all $x, x' \in [r, y]$. In other words, the thresholds are said to be monotone decreasing with respect to $[r, y]$ if alternatives further away from r have a weakly lower threshold. For example, for the tree illustrated in Fig. 1, on the path $[a, e]$ the thresholds, $q^{[a,e]}_a = n$, $q^{[a,e]}_c = n - 1$, $q^{[a,e]}_d = n - 3$, $q^{[a,e]}_e = n - 3$ is a set of monotone decreasing thresholds for any n number of voters $n \geq 4$. A similar set of thresholds can be defined for the path $[a, b]$.

For any extremal path $[r, y] \in \mathbf{E}$ we define a complete strict ordering $<_r$ on $[r, y]$ as follows: $x <_r x'$ if and only if $x \in [r, x']$ for all distinct $x, x' \in [r, y]$. In other words, $x < x'$ if and only if x is strictly 'closer' to r in the path $[r, y]$. For any $x, x' \in [r, y]$ we say that $x \leq_r x'$ if either $x = x'$ or $x <_r x'$. We define a single-peaked strict pre-ordering with peak $<_r$ as a transitive binary relation which is single-peaked on every extremal path $[r, y] \in \mathbf{E}_r$. For example, for the given tree T in Fig. 1, the following single-peaked pre-ordering $<_a$ can be defined: (i) on the path $[a, b]$: $a <_a c <_a b$, and (ii) on the path $[a, e]$: $a <_a c <_a d <_a e$. Note that the pre-ordering need not compare alternatives in a path $[r, y]$ with those in $[r, y']$ for some distinct $[r, y], [r, y'] \in \mathbf{E}_r$.

Let n_x the number of voters who have x at their top, and let $\mathcal{S}(T_{[r,y]})$ denote the set of all single-peaked preferences defined over $[r, y]$ according to $<_r$. Next we define a q-threshold rule on T. For any terminal node, $r \in X$, let $\mathbf{E}_r$ denote the set of all extremal paths with r as an terminal node.

Definition 1 (**q-threshold rule on T**) *An s.c.f. $f_r^q : \bigcup_{S\in\mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ is a q-threshold rule on T with respect to $r \in X$ if there exists an single-peaked strict pre-ordering with peak $<_r$*[8] *and thresholds $q : X \times \mathbf{E}_r \to N$ which are (i) monotone decreasing according to $<_r$ on any extremal path $[r, y] \in \mathbf{E}_r$ and (ii) for all $x \in T$ such that deg$(x) \geq 3$, for all distinct extremal paths $[r, y], [r, y'] \in \mathbf{E}_r$ such that $x \in [r, y] \cap [r, y']$, we have $q_x^{[r,y]} + q_x^{[r,y']} < n + 2$. Moreover, for all $\pi \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$,*

$$f_r^q(\pi) = f_r^q(\pi_{[r,y]}) = x^*(\pi) = \underset{x\in Range(\pi_{[r,y]})}{\arg\min} \left(\sum_{l\leq_r x} n_l \geq q_x^{[r,y]} \right),$$

for all $[r, y] \in \mathbf{E}_r$ such that $x \in [r, y]$.

Therefore, a q-threshold rule with respect to a terminal node r define monotone decreasing thresholds on every extremal path $[r, y]$ which satisfy the property that for any alternative or node which lies in the intersection of two distinct extremal paths $[r, y]$ and $[r, y']$ should have the sum of its thresholds in these paths to be strictly than $n+2$. The rule then picks that alternative which is the smallest alternative according to $<_r$ which has more aggregate votes (votes of alternatives at the top which precede it) when the profile is restricted to any $[r, y]$. Note that by defining the single-peaked pre-ordering $<_r$ on the whole tree we are defining an ordering on every extremal path $[r, y] \in \mathbf{E}_r$ with respect to which the set of threshold $q_x^{[r,y]}$ are monotone decreasing for any $x \in [r, y]$. We provide an example to illustrate the rule.

Example 2 *Consider the set of alternatives $X = \{a, b, c, d, e\}$, set of voters $N = \{1, 2, ..., 5\}$ and the undirected tree T as illustrated in Fig. 1. We will define a q-threshold rule f_a^q using $a \in X$. Consider the following set of monotone decreasing thresholds on every extremal path in $\mathbf{E}_a$:*

(i) For the extremal path $[a, e] \in \mathbf{E}_a$: $q_a^{[a,e]} = 5$, $q_c^{[a,e]} = 4$ $q_d^{[a,e]} = 2$ and $q_e^{[a,e]} = 2$.
(ii) For the extremal path $[a, b] \in \mathbf{E}_a$: $q_a^{[a,b]} = 5$, $q_c^{[a,b]} = 2$ and $q_b^{[a,b]} = 2$,

Note that the property $q_c^{[a,b]} + q_c^{[a,e]} = 6 < n+2 = 5+2 = 7$ is satisfied for $c \in [a, b] \cap [a, e]$ and deg$(c) = 3$. This guarantees the existence of an alternative $x^*(\pi)$ as mentioned in the definition of the rule. Since the rule only takes into account the range of the profile, we only specify the top-ranked alternatives of voters in a profile. Let π be such that $\tau(\pi) = (a, b, c, e, e)$ be the top-ranked alternatives of the five voters respectively. We

[8] A single-peaked strict pre-ordering with peak a is a transitive binary relation which is single-peaked on every extremal path $[a, y] \in \mathbf{E}_a$.

first show the existence of an alternative $x^*(\pi)$ (we show that $x^*(\pi) = c$) as mentioned in the definition of the rule. To find $x^*(\pi)$ we restrict the profile to every extremal path which consists of a, i.e., $\mathbf{E}_a = \{[a, b], [a, e]\}$. Since the rule only considers the top-ranked alternatives we write $f_a^q(\pi)$ as $f_a^q(\tau(\pi))$. The following properties of f_a^q can be used to find $x^*(\pi)$:

(i) *Since d is the first alternative according to $<_a$ in $[a, e]$ which has more 'cumulative' votes (3), i.e., the top-votes for alternatives which precede it (weakly) according to $<_a$ in the path $[a, e]$, i.e., a and c) than its threshold, $q_d^{[a,e]} = 2$, we have,*

$$f_a^q(\pi_{[a,e]}) = f_a^q(a, c, c, e, e) = \underset{x \in Range(\pi_{[a,e]})}{\arg\min} \left(\sum_{l \leq_r x} n_l \geq q_x^{[a,e]} \right) = d.$$

Note that here we used the fact that the restriction of the profile π to the path $[a, e]$ is such that $\tau(\pi_{[a,e]}) = (a, b, c, e, e)$. Note that the restriction of the profile π to $[a, e]$, i.e., $\pi_{[a,e]} = (a, c, c, e, e)$ since the only alternative that is not available in $[a, e]$ is b. Therefore, when b is removed we can infer that voter 2's next preferred alternative will be c by single-peakedness on T. We use this reasoning throughout to compute restrictions of various profiles.

(ii) *On path $[a, b]$, c is the first alternative in $[a, c]$ which has more 'cumulative' votes (4), i.e., the top-votes for alternatives which precede c (weakly) according to $<_a$ in the path $[a, b]$, i.e., a and c) than its threshold, $q_c^{[a,b]} = 4$, we have,*

$$f_a^q(\pi_{[a,b]}) = f_a^q(a, b, c, c, c) = \underset{x \in Range(\pi_{[a,b]})}{\arg\min} \left(\sum_{l \leq_r x} n_l \geq q_x^{[a,b]} \right) = c.$$

Note that here we used the fact that the restriction of the profile π to the path $[a, b]$ is such that $\pi_{[a,b]} = (a, c, c, b, b)$. Also note that even though b has more cumulative votes than its threshold, since it succeeds c in the ordering $<_a$ it is not the smallest such alternative according to $<_a$.

We can now identify $x^*(\pi) = d$ for the given q-threshold rule, $f(\pi) = d$. Note that even though $f_a^q(\pi_{[a,b]}) = c$, it is not the outcome of the rule when the profile π is restricted to the path $[a, e]$ where c is still available. Therefore, the only alternative which is the outcome of the rule over the restriction of the profile to any extremal path which contains it is d.[9] *The condition $q_x^{[r,y]} + q_x^{[r,y']}$ guarantees that the term involving the argmin on the right-hand side of the definition of q-threshold rule is well-defined and unique.*

[9]Recall that d is not in $[a, b]$ so the condition continues to hold.

Therefore, an s.c.f. is a q-threshold rule on $T(E)$ for a given $r \in X$ consists of monotone decreasing thresholds on each extremal path $[r, y] \in \mathbf{E}_r$ which defines a unique alternative $x^*(\pi)$ for every $\pi \in \mathcal{S}(T)^n$ which is the smallest alternative (according to $<_r$) in any extremal path $[r, y]$ which receives more cumulative votes than its assigned threshold $q_x^{[r,y]}$. The existence of such an alternative for every profile is guaranteed by the property that for any two distinct extremal paths $[r, y]$ and $[r, y']$, $q_x^{[r,y]} + q_x^{[r,y']} < n + 2$ for all $x \in [r, y] \cap [r, y']$. Alternatively, these rules can also be defined using monotone increasing thresholds on extremal paths from r. We now show that the outcome of a q-threshold rule at any profile belongs to the range of the profile.

Claim 1 *Suppose f_r^q is a q-threshold rule. For every $\pi \in \mathcal{S}(T_S)^n$, then $f_r^q(\pi) \in$ Range(π) for all $\pi \in \mathcal{S}(T_S)$ for all $S \in \mathcal{P}(X)$.*

Proof. We first prove this for the case when $Range(\pi) \subseteq [r, y]$ for some $[r, y] \in \mathbf{E}_r$. Since $[r, y]$ is just a line, we define $\min(\pi)$ and $\max(\pi)$ according to $<_r$ as follows: $\min(\pi) = x \in Range(\pi)$ such that $x \leq_r y$ for all $y \in Range(\pi)$. Similarly, $\max(\pi)$ can be defined. Note that in this case, $Range(\pi) = [\min(\pi), \max(\pi)]$. Suppose for contradiction that $f_r^q(\pi) \notin Range(\pi)$ and assume w.l.o.g. that $f_r^q(\pi) >_r \max(\pi)$. By the definition of arg-min in part (i) of the above definition, $f_r^q(\pi)$ cannot lie outside the range since $\max(\pi)$ is an alternative that obtains as many cumulative votes as $f_r^q(\pi)$. But since $\max(\pi) <_r f_r^q(\pi)$, the latter cannot be the smallest alternative according to $<_r$ that meets the above condition. This is a contradiction. Therefore, $f_r^q(\pi) \leq_r \max(\pi)$. Similar arguments can be made to show that $f_r^q(\pi) \geq_r \min(\pi)$. Therefore, $f_r^q(\pi) \in Range(\pi)$ if $Range(\pi) \subseteq [r, y] \in \mathbf{E}_r$.

We now prove the above claim more generally. Suppose for a given profile $\pi \in \mathcal{S}(T)^n$, $Range(\pi) \not\subseteq [r, y]$ for any $[r, y] \in \mathbf{E}_r$, i.e., the range of π is not a subset of any extremal path $[r, y]$. For contradiction, suppose $f_r^q(\pi) \notin Range(\pi)$. Then there must exist an extremal path $[r, \bar{y}]$ such that $f_r^q(\pi) \in [r, \bar{y}]$ but $f_r^q(\pi) \notin [\min(\pi_{[r,\bar{y}]}), \max(\pi_{[r,\bar{y}]})]$. By the definition of q-threshold rule, the restriction of the profile to $[r, \bar{y}]$ will not change the outcome. Therefore, $f_r^q(\pi_{[r,\bar{y}]}) \notin [\min(\pi_{[r,\bar{y}]}), \max(\pi_{[r,\bar{y}]})] = Range(\pi_{[r,\bar{y}]})$. But this is a contradiction to the above observation and the definition of q-threshold rule that $f_r^q(\pi_{[r,\bar{y}]}) \in Range(\pi_{[r,\bar{y}]})$ when $f_r^q(\pi) \in [r, \bar{y}]$. ■

By the definition, the outcome of any q-threshold rule is the *unique* alternative (say, $x^*(\pi)$) in the range of the profile $\pi \in \mathcal{S}(T)^n$ which is also the outcome of the s.c.f. $f(\pi_{[r,y]})$ for all such profiles which are restrictions of the profile π to any extremal path $[r, y] \in \mathbf{E}_r$ given that $x \in [r, y]$. To argue this we show that such an alternative exists and is unique.

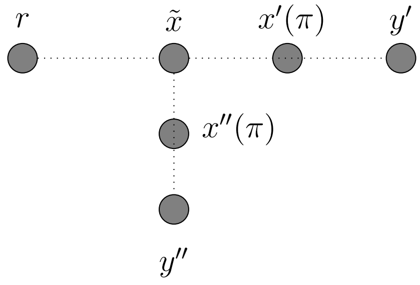

Figure 2: Illustration for Claim 2

Claim 2 *Suppose f^q_r is q-threshold rule on T. For every profile $\pi \in \mathcal{S}(T)^n$ there exists a unique alternative $x^*(\pi) \in Range(\pi)$, such that $f^q_r(\pi) = f^q_r(\pi_{[r,y]}) = x^*(\pi)$ for all $[r, y] \in \mathbf{E}_r$ such that $x^*(\pi) \in [r, y]$.*

Proof. We prove by contradiction. Suppose for a profile $\pi \in \mathcal{S}(T)^n$ there exist two distinct alternatives $x'(\pi)$ and $x''(\pi)$ which satisfy the property of $x^*(\pi)$ mentioned in the definition of q-threshold rule. Let $x'(\pi) \in [r, y']$ and $x''(\pi) \in [r, y'']$. If $[r, y'] = [r, y'']$ then we arrive at a contradiction immediately due to the definition of q-threshold rules, i.e., $f^q_r(\pi_{[r,y']}) = x'(\pi) = f^q_r(\pi_{[r,y'']}) = x''(\pi)$. Suppose $[r, y'] \neq [r, y'']$, then $[r, y'] \cap [r, y''] \neq \emptyset$. Pick the alternative in $[r, y'] \cap [r, y'']$ which furthest away from r; if no other alternative is available then pick r. Let this alternative be denoted as $\tilde{x}$. Note that $\tilde{x} \in [r, x'(\pi)] \subseteq [r, y']$ and $\tilde{x} \in [r, x''(\pi)] \subseteq [r, y'']$ by construction. Also note that both $x'(\pi), x''(\pi) \notin [r, y'] \cap [r, y'']$, otherwise we get a contradiction that $f^q_r(\pi_{[r,y']}) = x'(\pi) = f^q_r(\pi_{[r,y'']}) = x''(\pi)$. Similar arguments show that neither of them can be in $[r, y'] \cap [r, y'']$. This is illustrated in Fig. 2.

Suppose the profile is restricted to $S = \{\tilde{x}, x'(\pi), x''(\pi)\}$. Note that $\tau_i(\pi_S) \in \{\tilde{x}, x'(\pi), x''(\pi)\}$ for all $i \in N$. Let $e = \#\{i \in N : \tau_i(\pi_S) = \tilde{x}\}$, $c = \#\{i \in N : \tau_i(\pi_S) = x'(\pi)\}$ and $d = \#\{i \in N : \tau_i(\pi_S) = x''(\pi)\}$. By single-peakedness over T when π is restricted further to $S' = \{\tilde{x}, x'(\pi)\}$ the peaks at $x''(\pi)$ will be transferred to $\tilde{x}$ i.e. $[\tau_i(\pi_S) = x''(\pi)] \implies [\tau_i(\pi_{S'}) = \tilde{x}]$. Similarly, by single-peakedness over T when π is restricted to $S'' = \{\tilde{x}, x''(\pi)\}$, the peaks at $x'(\pi)$ will be transferred to $\tilde{x}$ i.e. $[\tau_i(\pi_S) = x'(\pi)] \implies [\tau_i(\pi_{S''}) = \tilde{x}]$.

By our assumption and the definition of q-threshold rule on $[r, y']$ and $[r, y'']$, $f(\pi_{\{\tilde{x},x'(\pi)\}}) = x'(\pi)$ and $f(\pi_{\{\tilde{x},x''(\pi)\}}) = x''(\pi)$. The following conditions hold due to the above assumptions,

$$q^{[r,y']}_{\tilde{x}} > e + d \text{ and } q^{[r,y'']}_{\tilde{x}} > e + c.$$

Adding the above two inequalities, we get $q^{[r,y']}_{\tilde{x}} + q^{[r,y'']}_{\tilde{x}} > 2e + d + c$. By part (iii) of the definition of q-threshold rule on T, and the fact that $e + d + c = n$, we get,

$$n + 2 > q_{\tilde{x}}^{[r,y']} + q_{\tilde{x}}^{[r,y'']} > n + e.$$

Note that the above inequalities can hold only if $e = 0$. This implies that $c + d = n$. The above observations imply that the following three expressions must hold,

$$n + 2 > q_{\tilde{x}}^{[r,y']} + q_{\tilde{x}}^{[r,y'']} > n, \quad q_{\tilde{x}}^{[r,y']} > d, \quad q_{\tilde{x}}^{[r,y'']} > c.$$

It is easy to verify that the above conditions cannot be met if $c + d = n$. This is a contradiction. Therefore, there is a unique $x^*(\pi)$ which satisfies the condition in the definition of q-threshold rule.

■

3 Axioms

Contraction consistency. An s.c.f. f is *contraction consistent* if for all $\pi \in \mathcal{S}(T)^n$ and for any $S' \in \mathcal{P}(X)$,

$$[f(\pi) \in S'] \implies [f(\pi) = f(\pi_{S'})].$$

Contraction consistency requires that the s.c.f produce the same outcome at π as the one it produces at any restriction of the profile to any subset S' containing $f(\pi)$. This axiom is a version of Sen (1977)'s 'contraction consistency' applied to social choice functions.

It is easy to check the dictator rule f^i is *contraction consistent.* Consider the following arguments: for any profile $\pi \in \mathcal{S}(T)^n$, we have $\tau_i(\pi) = \tau_i(\pi_S)$ for all $S \in \mathcal{P}(X)$ if $\tau_i(\pi) \in S$. Therefore, $f^i(\pi) = \tau_i(\pi) = f^i(\pi_S) = \tau_i(\pi_S)$. We require some standard axioms in addition to the above axiom for our main result.

Anonymity. An s.c.f. f satisfies *anonymity* if for all bijections $\sigma : N \to N$ and for all $\pi \in \mathcal{S}(T_S)^n$,

$$f(\pi) = f(\pi_\sigma) \quad \text{for all } S \in \mathcal{P}(X),$$

where $\pi_\sigma = (\pi_{\sigma(1)}, ..., \pi_{\sigma(n)})$ is the profile of permuted preferences. *Anonymity* states that permuting the preferences of voters does not change the outcome.

Unanimity. An s.c.f. f satisfies *unanimity* if for all $\pi \in \mathcal{S}(T_S)^n$ such that $\tau_i(\pi) =$

$a \in X$ for all $i \in N$, then

$$f(\pi) = a \quad \text{for all } S \in \mathcal{P}(X).$$

Unanimity requires that when every voter has the same peak then the outcome must be the peak. Our first result shows that any s.c.f. that is *contraction consistent* and unanimous must be tops-only.

4 Results

Proposition 1 *If an s.c.f.* $f : \bigcup_{S\in\mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ *is contraction consistent and unanimous then it is tops-only.*

Proof. See Appendix.

Therefore, the outcome of an s.c.f. which is *consistent* and *unanimous* depends only on the peaks of voters. An implication of this is that $f(\pi) \in Range(\pi)$ for all $\pi \in \mathcal{S}(T_S)^n$ for any $S \in \mathcal{P}(X)$. Suppose for contradiction that it is not the case and that for some $\pi \in \mathcal{S}(T)^n$, $f(\pi) \notin Range(\pi)$. This implies that for some $\hat{i} \in N$, $\tau_{\hat{i}} \in [a, b] \in \mathbf{E}$ and $\tau_{\hat{i}}(\pi) \in [a, f(\pi)]$. Consider $S = \{\tau_i, f(\pi)\}$. By single-peakedness over T, we get $f(\pi_S) = f(\tau_{\hat{i}}(\pi), ..., \tau_{\hat{i}}(\pi))$ since $\tau_{\hat{i}}(\pi) \succ_i f(\pi)$ for all $i \in N$. By unanimity, $f(\pi_S) = \tau_{\hat{i}}(\pi)$. By contraction consistency, $f(\pi) = f(\pi_S) = \tau_{\hat{i}}(\pi) \in Range(\pi)$. This is a contradiction. Therefore, it must be the case that $f(\pi) \in Range(\pi)$ for all $\pi \in \mathcal{S}(T_S)^n$ for all $S \in \mathcal{P}(X)$.

Theorem 1 *An s.c.f.* $f : \bigcup_{S\in\mathcal{P}(X)} \mathcal{S}(T_S)^n \to X$ *is contraction consistent, unanimous and anonymous if and only if it is a q-threshold rule on* T*.*

Proof. See Appendix.

Theorem 1 provides a characterization of *contraction consistent* voting rules. The proof of the result relies on Proposition 1. A property of tops-only rules that is used frequently in the proof is that when a profile over X is restricted to a subset of X, this acts as a 'new' profile over X with a smaller set of top-ranked alternatives. Moreover, the same set of top-ranked preferences can be generated by different profiles. These two properties give the main axiom- contraction consistency, a lot of bite when characterizing q-threshold rules. We provide a basic outline of the formal proof.

We first show that these s.c.f.s are *q-threshold rules* on any path $[r, y] \in \mathbf{E}$. We fix a terminal node $r \in X$ and define q-threshold rules on any path $[r, y] \in \mathbf{E}_r$. By Proposition 1 any s.c.f which is unanimous and *contraction consistent* must be *tops-only*. This implies that any restriction of the rule to a path $[r, y]$ must pick an alternative in the range of the restricted profile. To ensure that restrictions of the rule to different extremal paths in $\mathbf{E}_r$ do not contradict each other, another property, called *intersectionality* is required. It states that if the outcome of a profile $\pi \in \mathcal{S}(T)^n$ when restricted to an extremal path $[r, y] \in \mathbf{E}_r$ does not lie in $[r, y] \cap [r, y']$ for another extremal path $[r, y'] \in \mathbf{E}_r$, then the outcome of π when restricted to $[r, y']$ must lie in $[r, y] \cap [r, y']$. An implication of this property is that the sum of thresholds of an alternative which has degree greater than or equal to three on two distinct extremal paths $[r, y]$ and $[r, y']$ in $\mathbf{E}_r$ must be less than or equal to $n + 1$. This completes the characterization of the rule on T using the rules defined on every extremal path $[r, y]$ in $\mathbf{E}_r$.

5 Conclusion

This paper characterizes the class of *contraction consistent* social choice functions in the single-peaked domain over trees. The s.c.f.s we characterize, *q-threshold rules on trees*, can be seen as generalized versions of positional rules such as the min, max and median s.c.f.s when restricted to a line.

6 Appendix

Proof of Proposition 1 We argue that we only need to prove the claim for all $\pi \in \mathcal{S}(T)^n$. By *contraction consistency*, $f(\pi_S)$ for any $\pi_S \in \mathcal{S}(T_S)$ for some $S \in \mathcal{P}(X)$ will be invariant to changes in the 'tops' of restricted profiles. We first show that $f(\pi) \in Range(\pi)$ for any $\pi \in \mathcal{S}(T)^n$. Suppose for contradiction that $f(\pi) \notin Range(\pi)$. Take an alternative $x^*(\pi) \in X$ which is closest to $f(\pi)$ in T and also in the range of π, i.e., $x^*(\pi) \in Range(\pi) \cap [\tau_k(\pi), f(\pi)]$ for some voter $k \in N$ such that there is no other $x' \in [x^*(\pi), f(\pi)] \cap Range(\pi)$. By single-peakedness over a tree, since $x^*(\pi) \in [\tau_i(\pi), f(\pi)]$, we have $x^*(\pi) \succ_i f(\pi)$ for all $i \in N$. By unanimity, for $S = \{x^*(\pi), f(\pi)\}$ we have $f(\pi_S) = f(x^*(\pi), ..., x^*(\pi)) = x^*(\pi)$. By *contraction consistency*, we have $f(\pi) = f(\pi_S)$. This is a contradiction since $f(\pi) \neq x^*(\pi)$. Therefore, $f(\pi) \in Range(\pi)$ for all $\pi \in S(T)^n$.

Figure 3: Proving tops-onlyness

We now prove the tops-only property. Let $\pi = (\succ_i)_{i \in N}$ and $\pi' = (\succ'_i)_{i \in N}$ such that $\tau(\pi) = \tau(\pi')$. We show that $f(\pi) = f(\pi')$. Suppose for contradiction that $f(\pi) \neq f(\pi')$. Let $f(\pi)^+$ be the alternative adjacent to $f(\pi)$ and lies in the path $[f(\pi), f(\pi')]$ (shown in Figure 2).

We construct a profile $\hat{\pi} \in \mathcal{S}(T)^n$ by changing voter preferences in π such that,

$$\hat{\pi}_{\{f(\pi),f(\pi)^+\}} = \pi_{\{f(\pi),f(\pi)^+\}} \text{ and } \hat{\pi}_{\{f(\pi),f(\pi')\}} = \pi'_{\{f(\pi),f(\pi')\}}. \tag{$*$}$$

There are three types of voters in π and π' whose preferences we change sequentially as follows.

Case 1: Consider a voter $i \in N$, such that $f(\pi) \in [\tau_i(\pi), f(\pi)^+]$ or $f(\pi) \in [\tau_i(\pi'), f(\pi)^+]$ (since $\tau(\pi) = \tau(\pi')$). By single-peakedness, voter i prefers $f(\pi)$ to $f(\pi)^+$. Since $\tau(\pi) = \tau(\pi')$, these voters have the same top in π' as well. Therefore, by single-peakedness, $f(\pi) \in [\tau_i(\pi'), f(\pi)^+]$ implies that $f(\pi) \succ'_i f(\pi')$. We bring $f(\pi)$ to the top of the preferences of these voters. All the alternatives $x \in [\tau_i(\pi), f(\pi)]$ can be moved below the peak but above the alternatives to the left of x as we move further away from $f(\pi)$. Therefore, for any voter $i \in N$, we make the following changes:

(i) If $f(\pi) \in [\tau_i(\pi), f(\pi)^+]$ then $\tau_i(\hat{\pi}) = f(\pi)$.

(ii) For all $x, y \in X, x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$, then $x \hat{\succ}_i y$.

All the other alternatives are adjusted accordingly as per the definition of single-peakedness as we move away from the peak, $\tau_i(\hat{\pi}) = f(\pi)$. This ensures that for these voters the conditions in Equation $(*)$ are met.

Case 2: Consider any voter $i \in N$ such that $\tau_i(\pi) = \tau_i(\pi') \in [f(\pi)^+, f(\pi')]$.

By single-peakedness, $f(\pi)^+ \succ_i f(\pi)$ which is consistent with the first part of $*$. We bring $f(\pi)^+$ to the top of the preference. However, their preferences may not satisfy the second condition with respect to the preference profile π'. To account for this, we make the following changes, for any voter $i \in N$,

(i) If $\tau_i(\pi) \in [f(\pi)^+, f(\pi')]$ then $\tau_i(\pi) = f(\pi)^+$. Moreover, if $f(\pi) \succ'_i f(\pi')$ then $f(\pi) \hat{\succ}_i f(\pi')$, otherwise, if $f(\pi') \succ'_i f(\pi)$ then $f(\pi') \hat{\succ}_i f(\pi)$.

(ii) For all $x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$, then $x \hat{\succ}_i y$.

Condition (i) above ensures that both parts of the Equation $(*)$ are satisfied with respect to the given alternatives, while condition (ii) ensures that the new preference, $\hat{\succ}_i$, is single-peaked with respect to all the alternatives.

Case 3: Consider any voter $i \in N$ such that $f(\pi') \in [f(\pi)^+, \tau_i(\pi)]$. All these voters will have the same preferences over the pairs $\{f(\pi), f(\pi)^+\}$ and $\{f(\pi), f(\pi')\}$ due to single-peakedness. Therefore, for these voters both the conditions in Equation $(*)$

are satisfied and no further change is required. Similar to Case 1, we change the preferences as follows:

(i) If $f(\pi') \in [\tau_i(\pi)), f(\pi)^+]$ then $\tau_i(\hat{\pi}) = f(\pi')$.

(ii) For all $x, y \in X, x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$ then $x \mathrel{\hat{\succ}_i} y$.

Therefore, the other alternatives are adjusted accordingly as per the definition of single-peakedness as we move away from the peak, $\tau_i(\hat{\pi}) = f(\pi)$. This ensures that the conditions in Equation $(*)$ are met for these voters.

Case 4: Consider any voter $i \in N$ for which none of the above conditions are satisfied. This implies that $\tau_i(\pi) \notin [f(\pi)^+, f(\pi')]$, $f(\pi)^+ \in [f(\pi), \tau_i(\pi)]$ and $f(\pi') \notin [\tau_i(\pi), f(\pi)^+]$. In other words, these voters have peaks which lie in one of the 'branches' of the tree T emanating from alternatives which lie in the path $[f(\pi)^+, f(\pi')]$ excluding $f(\pi)^+$ and $f(\pi')$. These voters satisfy the first part of Equation $(*)$ but may not satisfy the second part.

(i) If $\tau_i(\pi) \notin [f(\pi)^+, f(\pi')]$, $f(\pi)^+ \in [f(\pi), \tau_i(\pi)]$ and $f(\pi') \notin [\tau_i(\pi), f(\pi)^+]$ then let $[\tau_i(\hat{\pi}) = f(\pi)^+]$. Moreover, if $f(\pi) \succ_i' f(\pi')$ then $f(\pi) \hat{\succ}_i f(\pi)'$, otherwise, if $f(\pi') \succ_i' f(\pi)$ then $f(\pi') \hat{\succ}_i f(\pi)$.

(ii) Moreover, for all $x \neq y$ if $x \in [\tau_i(\hat{\pi}), y]$ then $x \mathrel{\hat{\succ}_i} y$.

The changes above ensure that $\hat{\pi}$ satisfies the conditions in Equation $(*)$ with respect to both pairs of alternatives and is single-peaked.

Step 2: In Step 1, we constructed another profile $\hat{\pi} \in \mathcal{S}(T)^n$ from π and π' which satisfies Equation $(*)$. Due to the above arguments, $f(\hat{\pi}) \in Range(\hat{\pi})$, which implies that $f(\hat{\pi}) \in \{f(\pi), f(\pi)^+\}$. If $f(\hat{\pi}) = f(\pi)^+$, then by *contraction consistency*, we have $f(\hat{\pi}) = f(\hat{\pi}_{\{f(\pi), f(\pi)^+\}}) = f(\pi_{\{f(\pi), f(\pi)^+\}}) = f(\pi)$, where the second inequality is due to Equation $(*)$ in our construction and the last inequality is an implication of *contraction consistency*. But this is a contradiction since $f(\pi) \neq f(\pi)^+$. Therefore, $f(\hat{\pi}) = f(\pi)$.

By *contraction consistency* and the above observation,

$$f(\hat{\pi}) = f(\hat{\pi}_{\{f(\pi), f(\pi')\}}) = f(\pi_{\{f(\pi), f(\pi')\}}) = f(\pi').$$

The above two equations imply that $f(\pi) = f(\pi')$. This is a contradiction. Therefore, $f(\pi) = f(\pi')$. ■

Proof of Theorem 1 We prove necessity of the axioms first. It is easy to check that q-threshold rules on T are anonymous and unanimous. We show that q-threshold rules are *contraction consistent*. We prove this property for any profile $\pi \in \mathcal{S}(T)^n$.

By property of the q-threshold rule, there exists an $x^*(\pi) \in X$ for every given profile such that $f_r^q|_{[r,y]}(\pi_{[r,y]}) = x^*(\pi)$ for all $[r,y] \in \mathbf{E}_r$. Suppose for contradiction that $f(\pi) = x^*(\pi)$ but for some $S \in \mathcal{P}(X)$, $f(\pi_S) = x' \neq x^*(\pi)$. Take $S' = \{x', x^*(\pi)\} \subset [r,y']$ for some $[r,y'] \in \mathbf{E}_r$. By the definition of the rule,

$$f_r^q(\pi) = f_r^q(\pi_{[r,y']}) = x^*(\pi) = \underset{x \in Range(\pi_{[r,y']})}{\arg\min} \left(\sum_{l \leq_r x} n_l \geq q_x^{[r,y']} \right), \text{ and}$$

$$f_r^q(\pi_S) = f_r^q((\pi_S)_{[r,y']}) = x' = \underset{x \in Range(\pi_{[r,y']})}{\arg\min} \left(\sum_{l \leq_r x} n_l \geq q_x^{[r,y']} \right).$$

However, this is a contradiction since $\pi_{[r,y']} = (\pi_S)_{[r,y']}$.

We prove sufficiency first. Suppose f is an s.c.f. on T which is *contraction consistent*, *unanimous* and *anonymity*. By Proposition 1, f is tops-only so we will only need to keep track of the changes in the top of the preferences in a profile. We fix an alternative on a terminal node, say r. Let $f|_{[r,y]}$ be the restriction of the function f to all profiles which have top-ranked alternatives in the extremal path $[r,y] \in \mathbf{E}_r$ i.e. $f(\pi) = f|_{[r,y]}(\pi)$ for all π such that $\tau(\pi) \subseteq [r,y]$. We first show that $f|_{[r,y]}$ is a q-threshold rule on any extremal path $[r,y] \in \mathbf{E}_r$ i.e. for all $\pi \in \mathcal{S}(T)^n$ such that $\tau(\pi) \subseteq [r,y]$,

$$f_{[r,y]}(\pi) = f(\pi) = f(\pi_{[r,y]}) = x^*(\pi) = \underset{x \in Range(\pi_{[r,y]})}{\arg\min} \left(\sum_{l \leq_r x} n_l \geq q_x^{[r,y]} \right)$$

for some set of monotone decreasing thresholds $q_x^{[r,y]} : [r,y] \to N$. By Proposition 1 we know that $f|_{[r,y]}$ will also be a tops-only rule. We show that it must be a q-threshold rule with monotone decreasing thresholds $q_x^{[r,y]}$ on the path $[r,y]$ for any $x \in X$. We set the following ordering over the set of alternatives $[r,y]$: $x \leq_r y$ if and only if $x \in [r,y]$. For any $x \in [r,y]$, will denote as x^- and x^+ the alternatives which are adjacent before and after x respectively on the path $[r,y]$ i.e. $[r,y]$ is the sequence of alternatives $(r, ..., x^-, x, x^+, ..., y)$. Let the thresholds be defined as follows: for all $x \in [r, y^-]$,

$$q_x^{[r,y]} = \underset{q \in N}{\arg\min} \left(f|_{[r,y]}(\underbrace{x, ..., x}_{q \text{ votes}}, \underbrace{x^+, ..., x^+}_{n-q \text{ votes}}) = x \right),$$

i.e. $q_x^{[r,y]}$ is the minimum votes required at the top for x to beat the next alternative towards y on the path $[r,y]$. For y, let $q_y^{[r,y]} = 1$. We prove a stronger version of this claim next.

Claim 3 *For any* $x \in [r,y]$ *with threshold* $q_x^{[r,y]}$, $f|_{[r,y]}(\underbrace{x,...,x}_{k \text{ votes}}, \underbrace{\mathbf{x}'}_{n-k \text{ votes}}) = x$ *for all* $k \in \{q_x^{[r,y]},...,n\}$, *for all* $\mathbf{x}' \in [x^+, y]^{n-k}$.

Proof: Consider the profile π such that $\tau(\pi) = (\underbrace{x,...,x}_{q_x^{[r,y]} \text{ votes}}, \underbrace{x^+,...,x^+}_{n-q_x^{[r,y]} \text{ votes}})$ and for every voter i such that $\tau_i(\pi) \neq x$, let their second ranked alternative be $\mathbf{x}'_i$ which is the i^{th} component of $\mathbf{x}$ as given in the statement of the claim. By definition of the threshold of x,

$$f(\pi) = f|_{[r,y]}(\underbrace{x,...,x}_{q_x^{[r,y]} \text{ votes}}, \underbrace{x^+,...,x^+}_{n-q_x^{[r,y]} \text{ votes}}) = x.$$

Consider $S = \{x, \mathbf{x}'\}$ where we abuse notation slightly to denote $\mathbf{x}'$ as the set of alternatives in the array $\mathbf{x}'$. Note that $\tau(\pi_S) \subseteq [r,y]$. By contraction consistency,

$$f(\pi) = f(\pi_S) = f|_{[r,y]}(\pi_S) = f(\underbrace{x,...,x}_{k \text{ votes}}, \underbrace{\mathbf{x}'}_{n-k \text{ votes}}) = x$$

for any $k \in \{q_x^{[r,y]},...,n\}$.

Claim 4 *We now prove that the thresholds are monotonic decreasing over the path* $[r,y]$ *i.e.* $q_x^{[r,y]} \geq q_{x^+}^{[r,y]}$ *for all* $x \in [r, y^-]$.

Proof: We prove by contradiction. Suppose there exist $x, x^+ \in [r,y]$ such that $q_x^{[r,y]} < q_{x^+}^{[r,y]}$. By construction, $q_y^{[r,y]} = 1 \leq q_{x^+}^{[r,y]}$. Therefore, $x^+ < y$.

Since $x^+ < y$ there exists an alternative $(x^{++}) \in X$ (we denote x^{++} as the adjacent alternative on the right of x^+ towards y in $[r,y]$) such that $x < x^+ < x^{++} \leq y$. Consider a profile π such that $\tau(\pi) = ((x^+)^{q_x^{[r,y]}}, (x^{++})^{n-q_x^{[r,y]}})$. We show that $f(\pi) \notin \{x^+, x^{++}\}$ thus violating the property $f(\pi) \in Range(\pi)$.

Suppose $f(\pi) = x^+$. By definition of threshold for x we have $q_{x^+}^{[r,y]} \leq q_x^{[r,y]}$. This is a contradiction to our assumption that $q_{x^9i+}^{[r,y]} > q_x^{[r,y]}$. Therefore, $f(\pi) \neq x^+$. Suppose next that $f(\pi) = x^{++}$ and consider $S = \{x, x^{++}\}$. There exists a profile $\pi \in \mathcal{S}(T_S)^n$, such that for any $i \in N$, $[\tau_i(\pi) = x^+] \Rightarrow [\tau_i(\pi_S) = x]$. Note that, $[\tau_i(\pi) = x^{++}] \Rightarrow [\tau_i(\pi_S) = x^{++}]$. By *contraction consistency*,

$$f(\pi_S) = f(\underbrace{x,...,x}_{q_x^{[r,y]} \text{ votes}}, \underbrace{x^{++},...,x^{++}}_{n-q_x^{[r,y]} \text{ votes}}) = x^{++}.$$

But by Claim 3, $f(\pi_S) = x$ which is a contradiction. Therefore, $q_x^{[r,y]} \geq q_{x^+}^{[r,y]}$ for all $x \in [r, y^-]$. We now prove more generally, that $f|_{[r,y]}$ is a q-threshold rule on the path

$[r, y]$ with thresholds as defined above.

Claim 5 *Suppose $\pi \in \mathcal{S}(T)^n$ such that $\tau(\pi) \subseteq [r, y]$. Let $x^*(\pi)$ be as defined earlier in the proof. Then $f(\pi) = x^*(\pi)$.*

Proof: Let $x^*(\pi) = \arg \min_{x \in Range(\pi_{[r,y]})} \sum_{l \leq_r x} n_l \geq q_x^{[r,y]}$. It is easy to check that $x^*(\pi)$ exists and is unique for the given profile. We show that $f(\pi) = x^*(\pi)$. Suppose for contradiction that $f(\pi) = x' \neq x^*(\pi)$ for some $x' \in \tau(\pi) \subseteq [r, y]$.

We first argue that $x' > x^*(\pi)$. Suppose for contradiction that $x' < x^*(\pi)$. Take $S = \{x', x'^+\}$. By contraction consistency and single-peakedness on $[r, y]$, we get $f(\pi) = f(\pi_S) = ((x')^k, (x'^+)^{n-k}) = x'$. By definition of the thresholds, this implies that the threshold of x' must be less than k. Note that k is also the cumulative vote for x' in π. This is a contradiction to the fact that $x' < x^*(\pi)$ since $x^*(\pi)$ is unique smallest alternative which has more cumulative votes than its threshold.

Therefore, $x' > x^*(\pi)$. Now, consider $S' = \{x^*, x'\}$. Note that, $f(\pi) = f(\pi_{S'}) = ((x^*(\pi)^k, (x')^{n-k}) = x'$. However, by Claim 4, $f(\pi_{S'}) = x^*(\pi)$ since $k \geq q_{x^*(\pi)}^{[r,y]}$. This is a contradiction, therefore, $f(\pi) = x^*(\pi)$.

Therefore, $f|_{r,y}$ is a q-threshold rule on $[r, y]$. Let us denote this s.c.f. as $f_{[r,y]}^q$. By the above arguments, $f|_{[r,y]} = f_{[r,y]}^q$ for all $y \in X \setminus \{r\}$ such that $[r, y] \in \mathbf{E}_r$. An important property needs to be satisfied to ensure that the rules are consistent across different paths. We will call this property *intersectionality*, which is defined below.

Definition 2 *Suppose $f_{[a,b]}$ and $f_{[a,d]}$ are two q-threshold rules on two extremal paths $[a, b]$ and $[a, d]$ respectively. They are said to be* ***intersectional*** *if for any $\pi \in \mathcal{S}(T)^n$,*

$$\Big[f_{[a,b]}(\pi_{[a,b]}) = x' \notin \big([a, b] \cap [a, d]\big)\Big] \Rightarrow \Big[f_{[a,d]}(\pi_{[a,d]}) \in \big([a, b] \cap [a, d]\big)\Big].$$

Intersectionality states that at least one of the outcomes of the two s.c.f.s defined on their respective extremal paths must lie in the intersection of the two paths when the profile π is restricted to the relevant path.

Take any $\pi \in \mathcal{S}(T)^n$. We show that the every pair of restrictions of the rule f to extremal paths in $\mathbf{E}_r$ are *intersectional*. Suppose $[a, b]$ and $[a, d]$ are two distinct extremal paths. If $\tau(\pi_{[a,b]}) = \tau(\pi_{[a,d]})$, then by definition of restriction of a rule and by *tops-only* property, $f|_{[a,b]}(\pi_{[a,b]}) = f|_{[a,d]}(\pi_{[a,d]})$. In this case, our claim follows directly.

Suppose $\tau(\pi_{[a,b]}) \neq \tau(\pi_{[a,d]})$ and $f|_{[a,b]}(\pi_{[a,b]}) = x' \notin [a, b] \cap [a, d]$, and assume for contradiction that $f|_{[a,d]}(\pi_{[a,d]}) = x'' \notin [a, b] \cap [a, d]$. By the *tops-only* property, we know that $f|_{[a,d]}(\pi_{[a,d]}) \in Range(\pi_{[a,d]})$. Therefore, it must be that $x' \in [c, b]$ (as

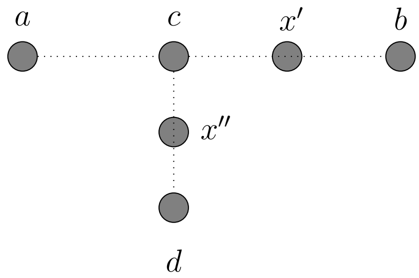

Figure 4: Illustration for Proof of Theorem 1 part (ii)

illustrated in Figure 3) where c is the last alternative away from a which is both in $[a, b]$ and $[a, d]$.

Now consider $\pi_{\{x',x''\}}$. Note that $Range(\pi_{\{x',x''\}}) = [x', x'']$ and $f|_{[d,b]}(\pi_{\{x',x''\}}) \in Range(\pi_{\{x',x''\}}) = [x', x'']$. Suppose $f|_{[d,b]}(\pi_{\{x',x''\}}) = \tilde{x}$. Then either $\tilde{x} \in [a, b]$ or $\tilde{x} \in [a, d]$. We show that $\tilde{x} \in \{x', x''\}$. Suppose for contradiction that $\tilde{x} \in [a, d] \setminus \{x', x''\}$. By *contraction consistency*, $f|_{[a,d]}(\pi_{[a,d]}) = x''$ implies that $f|_{[a,d]}(\pi_{\{\tilde{x},x''\}}) = x''$. This is a contradiction since $\tilde{x} \neq x''$. Similar contradiction is obtained if $\tilde{x} \in [a, b]$ Therefore, $f|_{[d,b]}(\pi_{\{x',x''\}}) \in \{x', x''\}$. Suppose $f|_{[d,b]}(\pi_{\{x',x''\}}) = x'$ without loss of generality.

We construct the following profile $\pi' = (\succ'_i)_{i\in N} \in \mathcal{S}(T)^n$ to obtain a contradiction to our initial assumption. Let $\tau_i(\pi') = \tau_i(\pi_{[a,b]}))$ for all $i \in N$. Also,

$$x' \succ_i x'' \implies x' \succ'_i x'' \text{ and } x'' \succ_i x' \implies x'' \succ'_i x'.$$

We argue that the above can be done without affecting the earlier step where we ensured that $\tau_i(\pi') = \tau_i(\pi_{[a,b]}))$ due to the following observations:

(a) Any voter who preferred x' over x'' in π must be such that either $x' \in [x'', \tau_i(\pi)]$, in which case $\tau_i(\pi_{[a,b]}) \in [x', b]$, or has a peak $\tau_i(\pi)$ such that $x' \notin [\tau_i(\pi), x'']$ in which case $\tau_i(\pi_{[a,b]}) \in [a, x')$. In either case, for these voters their tops will be unaffected in the previous step.

(b) Any voter who preferred x'' over x' in π must be such that either $x'' \in [\tau_i(\pi), x']$ in which case $\tau_i(\pi_{[a,b]}) = c$ or $x'' \notin [\tau_i(\pi), x']$. In either case, for these voters $\tau_i(\pi_{[a,b]}) \in [a, b]$. Therefore, for these voters too their tops will be unaffected in the previous step.

In all the cases, there is no constraint while constructing π' when ensuring that preferences of each voter over x' and x'' are the same as they were in π. Therefore, by our construction, $\pi_{\{x',x''\}} = \pi'_{\{x',x''\}}$.

Similarly, we construct another profile π'' such that $\tau_i(\pi'') = \tau_i(\pi_{[a,d]})$ for all $i \in N$. We also ensure that,

$$x' \succ_i x'' \implies x' \succ''_i x'' \text{ and } x'' \succ_i x' \implies x'' \succ''_i x'.$$

By *tops-onlyness*, we have $f|_{[a,b]}(\pi_{[a,b]}) = f(\pi') = x'$ and $f|_{[a,d]}(\pi_{[a,d]}) = f(\pi'') = x''$. By *contraction consistency*, we have $f(\pi) = f|_{[b,d]}(\pi'_{\{x',x''\}}) = x'$ and $f(\pi'') = f|_{[b,d]}(\pi'_{\{x',x''\}}) = x''$. This is a contradiction.

(iii) We show that *intersectionality* implies the following: for all $x \in T$ such that $deg(x) \geq 3$, $q_x^{[r,y]} + q_x^{[r,y']} < n+2$ for all extremal paths $[r,y], [r,y'] \in \mathbf{E}_r$.

We first prove for even number of voters. Suppose for contradiction that the above condition is violated. Then, there exists a node x which has degree greater than or equal to 3 which belongs to two distinct extremal paths $[r,y]$ and $[r,y']$ in $\mathbf{E}_r$, and $q_x^{[r,y]} + q_x^{[r,y']} \geq n+2$ as shown in Figure 6 below. This implies that there exists an integer $k \in \{1,2,...,n\}$ such that $q_x^{[r,y]} + q_x^{[r,y']} - (k+1) = n$.

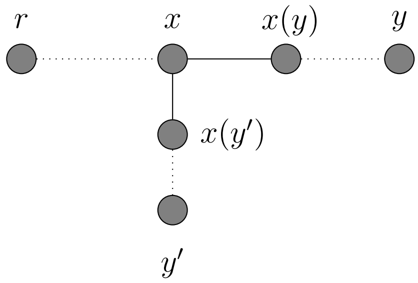

Figure 4

Consider the following profile with two types of preferences with the top three alternatives as follows:

$$\pi = \left[\begin{bmatrix} x(y') \\ x \\ x(y) \\ \vdots \end{bmatrix}^{q_x^{[r,y]}-1} \quad \begin{bmatrix} x(y) \\ x \\ x(y') \\ \vdots \end{bmatrix}^{q_x^{[r,y']}-k} \right]$$

where $x(y)$ and $x(y')$ are the alternatives adjacent to x away from r in the path $[r,y]$ and $[r,y']$ respectively and the preferences over other alternatives can be defined in any way consistent with single-peakedness. By definition of q-threshold rules on the path $[r,y]$ and $[r,y']$ we have,

$$f^q_{[r,y]}(\pi_{\{x,x(y)\}}) = f^q_{[r,y]}(x^{q_x^{[r,y]-1}}, x(y)^{q_x^{[r,y']}-k}) = x(y)$$

$$f^q_{[r,y]}(\pi_{\{x,x(y')\}}) = f^q_{[r,y']}(x^{q_x^{[r,y']}-k}, x(y')^{q_x^{[r,y]}-1}) = x(y')$$

The two equations are due to the fact that x does not have enough votes at the top to beat the other alternative since its threshold is strictly greater than its top votes. This is a contradiction to *intersectionality*. Therefore, $q_x^{[r,y]} + q_x^{[r,y']} < n + 2$ for any two paths $[r, y], [r, y'] \in \mathbf{E}_r$ such that $x \in [r, y] \cap [r, y']$. ■

References

Arrow, K. J., A. Sen, and K. Suzumura (2010): *Handbook of Social Choice & Welfare*, vol. 2, Elsevier.

Austen-Smith, D. and J. S. Banks (2005): *Positive Political Theory II: Strategy and Structure*, University of Michigan Press.

Bandyopadhyay, T. (1984): "On the frontier between possibility and impossibility theorems in social choice," *Journal of Economic Theory*, 32, 52–66.

Barberà, S. and M. O. Jackson (2004): "Choosing how to choose: Self-stable majority rules and constitutions," *The Quarterly Journal of Economics*, 1011–1048.

Bhattacharya, M. (2019): "Constitutionally consistent voting rules over single-peaked domains," *Social Choice and Welfare*, 52, 225–246.

Black, D., R. A. Newing, I. McLean, A. McMillan, and B. L. Monroe (1958): *The theory of committees and elections*, Springer.

Blair, D. H., G. Bordes, J. S. Kelly, and K. Suzumura (1976): "Impossibility theorems without collective rationality," *Journal of Economic Theory*, 13, 361–379.

Chernoff, H. (1954): "Rational selection of decision functions," *Econometrica: journal of the Econometric Society*, 422–443.

Demange, G. (1982): "Single-peaked orders on a tree," *Mathematical Social Sciences*, 3, 389–396.

Downs, A. (1957): *An Economic Theory of Democracy*, New York, Harper.

Hotelling, H. (1929): "Stability in Competition," *The Economic Journal*, 39, 41–57.

Koray, S. (2000): "Self-Selective Social Choice Functions Verify Arrow and Gibbard-Satterthwaite Theorems," *Econometrica*, 68, 981–996.

Koray, S. and A. Slinko (2008): "Self-selective social choice functions," *Social Choice and Welfare*, 31, 129–149.

Koray, S. and B. Unel (2003): "Characterization of self-selective social choice functions on the tops-only domain," *Social Choice and Welfare*, 20, 495–507.

Lainé, J., A. I. Ozkes, and R. Sanver (2016): "Hyper-stable social welfare functions," *Social Choice and Welfare*, 46, 157–182.

Nehring, K. and C. Puppe (2007): "The structure of strategy-proof social choice—Part I: General characterization and possibility results on median spaces," *Journal of Economic Theory*, 135, 269–305.

——— (2010): "Abstract arrowian aggregation," *Journal of Economic Theory*, 145, 467–494.

Schummer, J. and R. V. Vohra (2002): "Strategy-proof location on a network," *Journal of Economic Theory*, 104, 405–428.

Sen, A. (1969): "Quasi-transitivity, rational choice and collective decisions," *The Review of Economic Studies*, 36, 381–393.

——— (1977): "Social choice theory: A re-examination," *Econometrica: journal of the Econometric Society*, 53–89.