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Inequality in Hierarchies

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Abstract

The paper examines allocation of a resource in "hierarchies", where individuals are exogenously ordered and those ranked higher in the ordering make claim on the resource earlier. Individuals get strictly increasing and strictly concave payoff from resource extracted, giving rise to endogenous resource distribution. A "donor" supplies costly resource after the agents sequentially commit to resource shares. I characterize equilibrium resource distributions for a large class of donor's objectives (including utilitarian and egalitarian cases). When hierarchies are large, equilibrium resource shares follow a Geometric distribution over the ranks, irrespective of donor objective. With utilitarian (or egalitarian) donor, efficiency arises when payoffs are approximately linear.

JEL Codes: D30, D61, D63, Q15.
Keywords: Sequential access to resource, weak principal, Geometric distribution.

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I INTRODUCTION

Consider the problem of allocating water in a canal irrigation. Suppose farming plots of different farmers in a village are located in a straight line at various distances from the canal. This gives rise to a natural ordering among farmers. Once the irrigation department releases water in the canal, those who are nearer to the canal get to extract water earlier. Assume that each farmer is endowed with an identical, strictly increasing and strictly concave production function. Hence, the farmer nearest to the canal has an incentive to extract all the water. More generally, the farmers located higher in the ordering enjoy a positional advantage over those positioned below.[1] The irrigation department is aware of this issue, but can only control the aggregate water supply to discipline the farmers. Its objective is to maximize aggregate output of the village net of cost of water supply. The irrigation department may want to maximize aggregate output either because the bureaucrat's promotion is tied to performance, or the politician overseeing the irrigation department cares about votes which is increasing in aggregate output. Moreover, marginal cost of releasing water is positive for the government as it cares about sustainability of agriculture. I ask: does this allocation problem have a non-degenerate solution, i.e., one where the agent located at the front does not extract all the resources? And if yes, what is the nature of such equilibrium allocation? Specifically, how unequal is the distribution of water across farmers? And, are there conditions under which allocation is efficient?

There are many economic environments where similar allocation problems emerge. A foreign aid agency, interested in the aggregate welfare of a country (or motivated by its reputation among donor countries, which is increasing in aggregate welfare), needs to decide how much of costly aid to allocate to that country. Once aid is allocated, however, the head of the state first stakes claims on it, followed by other members of the elite connected to him, before it reaches the general citizens, positioned lowest in the ordering. A wealthy individual, while deciding to contribute to a research fund, needs to consider that once grant is allocated to a grant management body, research institutions get access to it sequentially in the order of some existing ranking among them (based on reputation or connection to the grant management body etc.). All research institutions have increasing and concave research production functions and the donor is interested in maximizing aggregate research output (i.e., aggregate impact of funding) net of funding cost. I refer to such economies as *hierarchies*.

I model a hierarchy as consisting of individuals with identical payoff functions defined over a resource and an exogenously given ordering among the individuals. Those ranked higher in the ordering get to make claim over the resource earlier. Importantly, the individual payoff function is strictly increasing and strictly concave. The individuals first sequentially commit to the shares of the resource that they plan to extract. There is a "donor" who supplies the resource at positive marginal cost after observing the sequence of commitments. In the benchmark case,

[1]This is a well-known problem in irrigation. See, for example, Bromley et al. (1980), Bardhan (1993), Jacoby and Mansuri (2020).

the donor's objective is to maximize the aggregate payoff of individuals net of cost.[2] The donor therefore behaves as a utilitarian central planner, albeit without its usual powers of commitment; she is not able to commit to any resource allocation strategy prior to the announcements by the individuals. The limited power of the donor makes the problem interesting. If the donor had that power, she could commit to allocating the efficient amount only when the agents announce equal shares and zero otherwise. This would have implemented the first best.

The donor therefore can be interpreted as a *weak principal* who can only discipline the behavior of individuals by changing the aggregate allocation of resource ex-post. The assumption about the donor being utilitarian is motivated by the leading examples above. An alternate interpretation for the assumption could be that the donor receives "kickbacks" from each agent which is proportional to the agent's payoff. Then, the problem is equivalent to the donor maximizing aggregate kickbacks. In an extension of the model, I examine the consequence of an egalitarian donor (and more general donor objectives) on equilibrium distribution of resource. In the example about foreign aid, if the aid agency's objective is to maximize the payoff of the least well-off member of the country, then it would behave as an egalitarian donor.

Hierarchies emerge because of the institutional or structural features of the economy. The ranks of the individuals in a hierarchy, therefore, need not necessarily reflect their power differences. In the context of irrigation, for example, the ranks of farmers is a function of their distance from the canal, which may or may not be correlated with their influence or power in the economy.[3] On the other hand, in the context of bureaucratic hierarchy, where bureaucrats can extract rents from citizens to deliver services, the ranks may indeed reflect power differences. Therefore, ranks in hierarchies can be interpreted more broadly than in the model of "jungle" economy described in Piccione and Rubinstein (2007). In a jungle economy, individuals, whose preferences admit bliss points, get sequential access to resources based on their ranks. The authors interpret ranks as reflecting power differences between individuals in a context without property rights. In hierarchies, property rights are respected, i.e., higher ranked individuals can not take away resource from lower ranked individuals. Moreover, preferences of individuals are "standard" allowing the model to comment on resource distribution and inequality.

A feature of hierarchical economies is that efficiency and equity are linked in this model. If the donor could choose individual allocations, she would have chosen equal share for everyone. In a hierarchy-less economy, where everyone is of same rank and hence, receives equal share of

[2] The individuals' ability to commit to their extraction strategies allows the model to be solved in a static framework. Similar assumptions about commitment is assumed in other agency problems involving separate timing of announcement and implementation of actions. For example, in (static) models of electoral competition, candidates (the agents) are assumed to have commitment power with respect to their announced platforms (Persson and Tabellini 2002, Chapters 3 and 4). Voters (the principal) observe the announced platforms and vote, knowing that the winning candidate would implement her announced policy. This assumption can be justified by considering a model where the principal and agents interact repeatedly and the principal can punish an agent for not following through on her announcement. It is similarly possible to implement the static equilibrium of this model in its dynamic version by deploying analogous punishment strategies.

[3] Jacoby and Mansuri (2020) for example find that in Pakistan, the distance of a farmer from canal is not correlated with their landholding.

the resource (by design), the donor will therefore allocate the efficient amount. In hierarchies, if the donor reduces aggregate allocation in response to unequal distribution, then equilibrium resource distribution may be associated with inefficient allocation. Characterization of the equilibrium distribution, therefore, may shed light on the degree of inefficiency that hierarchies generate, and factors that help shape it.

In the model, I restrict attention to payoff functions that exhibit constant relative risk aversion (CRRA) as it keeps the analysis tractable. I show that the model has a unique non-degenerate equilibrium when individuals' relative risk aversion parameter (ρ) is less than one, i.e., when payoff functions are less concave than the logarithm function. In equilibrium, all individuals extract positive shares of resource and the share falls monotonically with rank.[4] In large hierarchies, where the population is infinitely large and each individual has a different rank, the resource shares follow a Geometric distribution over the ranks of individuals, parameterized by ρ. Consequently, the ratio of equilibrium shares of any k^{th} and $(k+1)^{th}$ ranked individuals is constant and is given by $1/(1-\rho)$. Hierarchies, therefore, can engender significant inequality, especially when pay-off functions are sufficiently concave. Additionally, any policy that attempts to curb extraction of the some of the top ranked individuals would result in higher extraction by the rest of the individuals due to the memorylessness property of Geometric distribution.

An interesting observation about the analysis is that degree of relative risk aversion plays an important role in shaping resource inequality in hierarchies, even though the model is deterministic. This is because of two factors. First, equilibrium shares depend on how sensitive donor's optimal resource supply is to changes in shares. For a large enough share announced by any individual, donor's resource supply would fall with the announced share. Therefore, a more elastic supply reduces an individual's ability to extract higher share. Moreover, elasticity of resource supply depends on ρ. To understand why, notice that ρ measures the elasticity of marginal benefit of individual payoff w.r.t. resource extracted.[5] Since donor's marginal benefit of resource supply is a weighted average of individual marginal benefits, a higher ρ makes donor's marginal benefit more sensitive to resource extraction. If ρ is higher, therefore, in response to a higher share announced by any individual, the donor only needs to reduce her aggregate supply by a smaller amount to attain optimal supply. This, however, means that optimal supply is *less* elastic to shares when ρ is higher. This incentivizes the higher ranked individuals to extract more, due to the first factor, generating higher inequality. Conversely, when ρ is close to zero (i.e., approximate risk-neutrality), elasticity of resource supply becomes extremely large, almost eliminating higher ranked individuals' ability to extract more. This results in approximately equal shares for everyone and approximate efficiency in aggregate allocation.

In an extension of the model, in Section IV, I consider a resource "market" in hierarchies,

[4] When $\rho \geq 1$, equilibrium does not exist in this model.

[5] In other words, $\rho = -\frac{r\pi''(r)}{\pi'(r)} = -\frac{d \log \pi'(r)}{d \log r}$.

i.e., a context where several donors simultaneously supply the same resource. In case of foreign aid or research funding, it is natural to think of several suppliers of resource. I show that resource distribution becomes approximately efficient as number of donors becomes large. This happens because competition makes resource supply from *all donors* more sensitive to extraction strategies from any given donor. Extracting higher share from one donor can makes other donors to also reduce their supply, as the second donor's benefit of supplying resource would fall if the allocation is already highly unequal from the first donor. This limits higher ranked individual's power to extract more from any donor. Competition in supply, therefore, disciplines the demand side. This results in efficiency in the limit.

In another extension, I consider different objectives of the donor. I show that when donor's objective is egalitarian, the equilibrium resource distribution again depend on ρ. Specifically, for ρ small enough the shares are all equal, i.e., allocation is efficient. When ρ is large enough, the shares follow a Geometric distribution. Therefore, in finite hierarchies, the donor's objective can shape resource distribution. However, in large hierarchies, even with an egalitarian donor, the equilibrium distribution is Geometric, i.e., it is same as that under a utilitarian donor. Moreover, with a generalized utilitarian objective (i.e., with any arbitrary vector of welfare weights on individuals' payoffs), as well as with a generalized egalitarian objective (where welfare weight on the minimum payoff is positive but not necessarily one), the resource shares in large hierarchies are also characterized by the Geometric distribution with parameter ρ. Hence, for a large class of donor objectives, the inequality and welfare in large hierarchies are identical.

The literature on organization theory and team production have modeled hierarchies that serve various purposes in economic organizations; they shape organization of knowledge (Garicano 2000), assignment of monitoring responsibilities (Miller and Rozen 2014), facilitate delegation of authority (Melumad et al. 1995), limit rent seeking by supervisors (Mishra 2002) etc. In this model, hierarchies determine the sequence in which economic agents get access some productive resource. Additionally, while Garicano (2000) and Miller and Rozen (2014) endogenize formation of hierarchies, my model takes hierarchies as given and examines its consequence for allocation, as in the framework of Mishra (2002) and Mookherjee (2006). In the context of delegation of authority, risk-neutrality of agents down the hierarchy is one of the necessary and sufficient conditions for hierarchical delegation to be optimal (Mookherjee 2006). The result is similar to my model, where approximate risk-neutrality (of all agents) can generate efficiency, albeit for different reasons. The model therefore sheds a new light on how payoff concavity, even in a deterministic setup, can shape efficiency in hierarchies. The paper also contributes to our understanding of how and when hierarchical access to resources may engender efficiency. In models with finite demands of agents, allowing (randomized) sequential access to resources typically generates efficiency, as shown in Abdulkadiroğlu and Sönmez (1998). Piccione and Rubinstein (2007) also find that when individuals have finite demands, (deterministic) sequential access to resources maintains allocative efficiency in a "jungle" economy. They do however point out that when preferences are identical across individuals, equilibrium allocation entails higher ranked individuals's consumption bundle having higher value (computed using competitive equilibrium prices sustaining that allocation), i.e., resource allocation is unequal in value. My model shows that when distribution is endogenous, relative risk-aversion of individuals and competition in supply shape inequality in distribution. Hence, even with unrestricted demand, approximate efficiency is achievable. Some papers examine more directly the distributional consequences of hierarchies. Garicano and Rossi-Hansberg (2006), for example, show how knowledge based hierarchies in firms may result in unequal rewards across workers. Bothner et al. (2011), on the other hand, discuss how competition for status need not result in greater reward inequality, known as the Matthew effect.

II A Simple Case: $N = 2$

I illustrate the primary forces at work in the model by considering a hierarchical economy consisting of only two individuals, indexed $h = 1,2$. The individuals have identical payoff functions defined over a single resource R, given by $\pi(R)$. I assume that π is strictly increasing and strictly concave for all R and exhibits CRRA with parameter $\rho > 0$. I restrict attention to $\rho \in (0,1)$ since it ensures existence of equilibrium, as shown below. The individuals are positioned in hierarchy where individual 1 is ranked above 2; this implies that individual 1 first decides the fraction $\alpha \in [0,1]$ of the resource that he wishes to expropriate. Individual 2 therefore receives $(1-\alpha)$ fraction of the resource. The payoffs of the two individuals are $\pi(\alpha R)$ and $\pi((1-\alpha)R)$ respectively, where R is the aggregate supply of resource. In addition to the two individuals, there is a "donor" who decides the supply of R. The marginal cost to the donor of supplying R is positive and constant, given by c. The "donor" maximizes the sum of payoffs of the two individuals net of costs. The game proceeds as follows: first, individual 1 commits to $\alpha \in [0,1]$. The donor observes α and then chooses R. It is clear that in absence of the donor, i.e., when R is exogenous, individual 1 would always choose $\alpha = 1$. We examine the optimal choice of α in presence of the donor.

Let individual 1 choose some α. Then donor's optimization problem is:

$$\max_R \pi(\alpha R) + \pi((1-\alpha)R) - cR$$

The FOC gives us

$$\alpha\pi'(\alpha R) + (1-\alpha)\pi'((1-\alpha)R) = c \quad \text{for } \alpha \in (0,1) \tag{1}$$

$$\pi'(R) = c \quad \text{for } \alpha = 0,1$$

Let $R^*(\alpha)$ be the solution to the pair of equations above. It specifies the optimal resource supply by donor for any choice of α. By backward induction, individual 1 chooses α by taking into account donor's optimal response. Let α^* be the equilibrium fraction chosen by individual 1.

We then have the following result:

Lemma 1 *Equilibrium* $(\alpha^*, R^*(\alpha))$ *exists and is unique.* $\alpha^* \in (\frac{1}{2}, 1)$.

Proof: Existence and uniqueness of $R^*(\alpha)$ is given by the fact that $\pi^{'}(\cdot)$ is monotone and satisfies Inada conditions (since it is CRRA).
We have $R^*(1) = \pi^{'^{-1}}(c)$. Also, $R^*(\frac{1}{2})$ is given by

$$\frac{1}{2}\pi^{'}\left(\frac{1}{2}R^*(\frac{1}{2})\right) + \frac{1}{2}\pi^{'}\left(\frac{1}{2}R^*(\frac{1}{2})\right) \;=\; c$$

$$\Rightarrow \quad \frac{1}{2}R^*(\frac{1}{2}) \;=\; \pi^{'^{-1}}(c) \;=\; R^*(1)$$

Hence, individual 1 gets the same payoff from committing to $\alpha = \frac{1}{2}$ and $\alpha = 1$. Now, let's consider any $\alpha \in (\frac{1}{2}, 1)$. Then, $\alpha R^*(\alpha) > (1-\alpha)R^*(\alpha)$ which implies that $\pi^{'}(\alpha R^*(\alpha)) < \pi^{'}((1-\alpha)R^*(\alpha))$. Given equation (1), we get that

$$\pi^{'}(\alpha R^*(\alpha)) \;<\; c$$

$$\Rightarrow \quad \alpha R^*(\alpha) \;>\; R^*(1)$$

Hence, for all $\alpha \in (\frac{1}{2}, 1)$, individual 1's payoff is higher than $\alpha = 1$ and $\alpha = \frac{1}{2}$. By the same logic, for all $\alpha \in (0, \frac{1}{2})$, individual 1's payoff is lower. Also, $\alpha R^*(\alpha)$ is continuous in α at $\frac{1}{2}$. Differentiating equation (1) w.r.t. α we get,

$$\frac{\alpha}{R^*(\alpha)}\frac{dR^*(\alpha)}{d\alpha} \;=\; \frac{1-\rho}{\rho} \cdot \frac{c - \pi^{'}((1-\alpha)R^*(\alpha))}{c}$$

First, $\frac{dR^*(\alpha)}{d\alpha} < 0$ if $\alpha \in (\frac{1}{2}, 1)$ since $(1-\alpha)R^*(\alpha) < \pi^{'^{-1}}(c)$ and $\rho \in (0,1)$. Also, $\frac{dR^*(\alpha)}{d\alpha} = 0$ for $\alpha = \frac{1}{2}$. Hence, $\alpha R^*(\alpha)$ is upward sloping at $\frac{1}{2}$. Elasticity of $R^*(\alpha)$ w.r.t. α is given by

$$\xi \;=\; -\frac{\alpha}{R^*(\alpha)}\frac{dR^*(\alpha)}{d\alpha} \;=\; \frac{1-\rho}{\rho} \cdot \left(\frac{\pi^{'}((1-\alpha)R^*(\alpha))}{c} - 1\right) \tag{2}$$

At $\alpha = \frac{1}{2}$, $\xi = 0$. As α approaches 1, $(1-\alpha)R^*(\alpha)$ approaches zero. Hence $\xi \to \infty$. Therefore, there must exist a unique $\alpha^* \in (\frac{1}{2}, 1)$ such that $\xi = 1$ which maximizes individual 1's payoff. ■

The result shows that when $\rho \in (0,1)$, the problem posed above has a unique interior solution, i.e., the higher ranked individual does not expropriate all resources even though she would have preferred to do that. The allocation of resource, nonetheless, is inefficient as the higher ranked individual gets a strict majority share of the supplied resource. The proof of the result highlights a central tension in the model – individual 1 obviously wants to choose as high an α as possible, but the donor, being a utilitarian, does not prefer that. Consequently, when α is larger than half, the donor may respond to higher α by *lowering* R.

III A Model of Hierarchy

There are $N \geq 2$ individuals in the economy. $H \geq 1$ individuals have an exogenously given ordering or ranking, with ranks $h = 1, 2, \ldots, H$; $H \leq N-1$. The rest $N-H$ individuals have rank $H+1$, i.e., they are at the bottom of the hierarchy. All individuals have identical payoff functions defined over a single resource R. The payoff function defined over the resource is denoted by a strictly increasing and strictly concave CRRA function π with parameter $\rho \in (0,1)$. Each of the first H rank holding individuals sequentially commits to a fraction of the resource that they wish to consume. Let α_h for all $h \leq H$ be the share that rank holder h commits to. The sequence of commitments is given by the ranks, i.e., first rank 1 individual commits to α_1, followed by the rank 2 individual and so on. Let $\alpha_r = 1 - \sum_{h=1}^{H} \alpha_h$. The last $N-H$ individuals with rank $H+1$ divide the α_r residual fraction of the resource equally among themselves, i.e., $\alpha_{H+1} = \frac{\alpha_r}{N-H}$. They are the residual claimants in the economy. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_H, \alpha_{H+1})$. There is a donor who observes α and supplies R at a constant marginal cost $c > 0$. Donor's optimal choice $R^*(\alpha)$ is given by

$$R^*(\alpha) \equiv \operatorname*{argmax}_R U(R) \equiv \operatorname*{argmax}_R \frac{1}{N} \left\{ \sum_{h=1}^{H} \pi(\alpha_h R) + (N-H)\pi(\alpha_{H+1} R) - cR \right\}$$

$U(R)$ is the donor's objective, which is also the utilitarian welfare of the hierarchy for any given level of R, normalized by population. The normalization helps in examining welfare in large populations.

The tuple $\langle N, H, \pi, U \rangle$ denotes a hierarchy of order H with a utilitarian donor. $\langle N, 0, \pi, U \rangle$ denotes a hierarchy-less economy where all individuals shares the resource equally, in which case everyone receives $\pi^{'-1}(c)$ amount of resource. The welfare in the economy is given by $\bar{U} = \pi(\bar{R}) - c\bar{R}$ where $\bar{R} = \pi^{'-1}(c)$ is the efficient per capita resource allocation. This will serve as the benchmark in the analysis of a hierarchy of positive order (i.e., where $H \geq 1$).

Remark: As discussed in the Introduction, we can interpret the model as one where the donor is motivated by private payoffs. Suppose the donor's payoff is the sum of kickbacks she receives (in the unit of $\pi(R)$) from members of hierarchy after resource is allocated. Let λ denote the constant fraction of payoff that the donor demands as kickbacks from each individual. In such a setting, the donor would still be maximizing the same objective function as above except with a revised marginal cost given by c/λ. The objective functions of individuals also remain the same. Hence, as long as donor's private payoff is linear in the aggregate payoff of members of hierarchy, the analysis would remain unchanged. The requirement of linearity may seem unreasonable in the model with a single donor, as a privately motivated donor may benefit from optimally choosing the "price" (λ) of resource supply as a function of resource extraction strategies. This concern may be limited in the model with multiple donors, which we consider later in Section IV, where a "competitive" price of resource supply may emerge, making the case for linear payoffs more defendable.

III.I Equilibrium Characterization

Let's denote $a_h = \sum_{k=1}^{h} \alpha_k$ for $1 \leq h \leq H$ and $a_0 = 0$. Then the strategy for each individual h is to choose $\alpha_h \in [0, 1-a_{h-1}]$ after observing $(\alpha_1, \ldots, \alpha_{h-1})$. Alternately, h's strategy is to choose $\beta_h \in [0,1]$ where $\alpha_h = \beta_h(1-a_{h-1})$. Written this way gives the strategies a nested structure:

$$\begin{aligned}\alpha_h = \beta_h(1-a_{h-1}) &= \beta_h(1-a_{h-2}-\beta_{h-1}(1-a_{h-2})) \\ &= \beta_h(1-\beta_{h-1})(1-a_{h-2}) \\ &= \beta_h \prod_{j=1}^{k}(1-\beta_{h-j})(1-a_{h-k-1}) \quad \text{for any } k < h\end{aligned}$$

Similarly,

$$\alpha_{H+1} = \frac{1}{N-H}(1-a_H) = \frac{1}{N-H}(1-\beta_H)(1-a_{H-1}) = \frac{1}{N-H}\prod_{h=1}^{H}(1-\beta_h) \tag{3}$$

This structure of the strategies will become useful later to characterize the equilibrium and perform comparative statics. The solution concept used is Subgame Perfect Nash Equilibrium. Let $\alpha^* = (\alpha_1^*, \ldots, \alpha_H^*, \alpha_{H+1}^*)$ be the vector of equilibrium shares, given $R^*(\alpha)$.

Proposition 1 *In a hierarchy $\langle N, H, \pi, U\rangle$, equilibrium $(\alpha^*, R^*(\alpha))$ exists and is unique. $\alpha^* > 0$ and $\alpha_h^* > \alpha_{h+1}^*$ for all $h = 1, 2, \ldots, H$.*[6]

Proof: Donor's optimization implies, for any $\alpha > 0$,[7]

$$\sum_{h=1}^{H} \alpha_h \pi'(\alpha_h R^*(\alpha)) + (N-H)\alpha_{H+1}\pi'(\alpha_{H+1}R^*(\alpha)) = c \tag{4}$$

$R^*(\alpha)$ exists and is unique because LHS of equation (4) is monotonic in R (due to monotonicity of $\pi'(\cdot)$) and satisfies Inada conditions (because π' is CRRA).

Let's define $\Delta_h \equiv \pi'(\alpha_{H+1}R^*(\alpha)) - \pi'(\alpha_h R^*(\alpha))$ for all $h \leq H$. Now the elasticity of R^* w.r.t. α_H, given $(1-a_{H-1})$, is

$$\xi_H = \frac{1-\rho}{\rho c}\alpha_H \Delta_H$$

R is decreasing (increasing) in α_H when $\beta_H > (<) \frac{1}{N-H+1}$, since $\rho \in (0,1)$. Hence, optimal $\beta_H > \frac{1}{N-H+1}$. Also, ξ_H explodes to infinity as β_H approaches 1. Hence there exists a unique interior solution to α_H given by $\xi_H = 1$. Now, fix $(1-a_{H-2}) > 0$. Then for every $\alpha_{H-1} \in$

[6]For any vector x, $x > 0$ implies all elements of x are strictly positive, while $x \geq 0$ implies that all elements of x are non-negative with at least one strictly positive.

[7]The optimization condition for any $\alpha \geq 0$ can be defined accordingly.

$[0,1-a_{H-2})$, we know that there is a unique interior solution to α_H. Then the elasticity of R w.r.t. α_{H-1}, taking into account the optimal choice of α_H, is given by

$$\xi_{H-1} = \frac{1-\rho}{\rho c}\alpha_{H-1}\left[\Delta_{H-1} - \theta^H_{H-1}\Delta_H\right]$$

where $\theta^H_{H-1} = -\frac{d\alpha_H}{d\alpha_{H-1}} > 0$. Again, R is falling in α_{H-1} when β_{H-1} high enough and ξ_{H-1} explodes as β_{H-1} approaches one. Hence, we get a unique interior solution given by $\xi_{H-1} = 1$. By applying this logic recursively, we get that α^* exists, is unique and $\alpha^* > 0$. Additionally, $\alpha^* > 0$ solves, for all $h = 1,2,\ldots,H-1$,

$$\xi_h = \frac{1-\rho}{\rho c}\alpha^*_h\left[\Delta^*_h - \sum_{k=h+1}^{H}\theta^{k*}_h\Delta^*_k\right] = 1 \tag{5}$$

and

$$\xi_H = \frac{1-\rho}{\rho c}\alpha^*_H\Delta^*_H = 1 \tag{6}$$

where $\theta^{k*}_h = -\frac{d\alpha_k}{d\alpha_h}(\alpha^*) > 0$ and $\Delta^*_h = \Delta_h(\alpha^*)$. This implies,

$$\alpha^*_H\Delta^*_H = \frac{\rho c}{1-\rho}$$

$$\alpha^*_{H-1}\Delta^*_{H-1} = \frac{\rho c}{1-\rho}\left[1 + \theta^{H*}_{H-1}\frac{\alpha^*_{H-1}}{\alpha^*_H}\right]$$

The two equations above give us, $\alpha^*_{H-1}\Delta^*_{H-1} > \alpha^*_H\Delta^*_H$. Hence, we must have $\alpha^*_{H-1} > \alpha^*_H$. Otherwise, $\alpha^*_{H-1} \leq \alpha^*_H$ implies $\Delta^*_{H-1} \leq \Delta^*_H$, given the definition of Δ_h above, which is a contradiction. We now prove $\alpha^*_{h-1} > \alpha^*_h$ for $h < H$. Notice,

$$\begin{aligned}
\frac{d\alpha_h}{d\alpha_k} &= \frac{\partial\alpha_h}{\partial\alpha_k} + \sum_{l=1}^{h-k-1}\frac{\partial\alpha_{k+l}}{\partial\alpha_k}\times\frac{d\alpha_h}{d\alpha_{k+l}} \\
\Rightarrow \theta^h_k &= \theta^h_{k+1} + \frac{\partial\alpha_{k+1}}{\partial\alpha_k}\theta^h_{k+1} \quad \left(\text{since } \frac{\partial\alpha_{k+l}}{\partial\alpha_k} = \frac{\partial\alpha_{k+l}}{\partial\alpha_{k+1}} \text{ for any } l > 1\right) \\
\Rightarrow \theta^h_k &= (1-\theta^{k+1}_k)\theta^h_{k+1} \quad \left(\text{since } \frac{\partial\alpha_{k+1}}{\partial\alpha_k} = \frac{d\alpha_{k+1}}{d\alpha_k}\right) \qquad (7)
\end{aligned}$$

The optimality of α^*_{h-1} is given by

$$\alpha^*_{h-1}\left[\Delta^*_{h-1} - \sum_{k=h}^{H}\theta^{k*}_{h-1}\Delta^*_k\right] = \frac{\rho c}{1-\rho} \tag{8}$$

Now,

$$\begin{aligned}
\sum_{k=h}^{H} \theta_{h-1}^{k*}\Delta_k^* &= \sum_{k=h+1}^{H} \theta_{h-1}^{k*}\Delta_k^* + \theta_{h-1}^{h*}\Delta_h^* \\
&= (1-\theta_{h-1}^{h*}) \sum_{k=h+1}^{H} \theta_h^{k*}\Delta_k^* + \theta_{h-1}^{h*}\Delta_h^* \quad \text{(from equation (7))} \\
&> \sum_{k=h+1}^{H} \theta_h^{k*}\Delta_k^* \qquad (9)
\end{aligned}$$

where the last inequality follows from the fact that $\theta_{h-1}^{h*} \in (0,1)$ and $\alpha_h^* > 0$, i.e.,

$$\Delta_h^* > \sum_{k=h+1}^{H} \theta_h^{k*}\Delta_k^*.$$

Now if we compare equation (8) with the following equation for the optimality of α_h^*,

$$\alpha_h^* \left[\Delta_h^* - \sum_{k=h+1}^{H} \theta_h^{k*}\Delta_k^* \right] = \frac{\rho c}{1-\rho} \qquad (10)$$

we get that we must have $\alpha_{h-1}^* > \alpha_h^*$. Otherwise, $\alpha_{h-1}^* \leq \alpha_h^*$ implies $\Delta_{h-1}^* \leq \Delta_h^*$, which together with inequality (9) imply that the LHS of (8) is less than the LHS of (10) – a contradiction. ■

The proposition provides a generalization of Lemma 1. We find that there is an early mover advantage in the hierarchy, i.e., those ranked higher claim a higher share of resource. I briefly discuss the intuition behind the result. First, I observe that if $\rho \in (0,1)$, then for any α_h, R falls with α_h, for α_h high enough. This is because of the utilitarian nature of the donor. This implies that α_h^* must be in the falling region of R. Additionally, with higher α_h, the elasticity of R w.r.t. α_h, denoted by ξ_h, explodes to infinity, which ensures $\alpha^* \gg 0$. The interior solution is given by the set of equations $\xi_h = 1$ for all h. Now, changes in α_h would affect R directly as well as indirectly by changing α_k for all $k > h$. Since higher value of α_h would reduce α_k, the indirect effect attenuates some of the direct effect. For higher ranked individuals (i.e., those with low h), the indirect effect is higher since there are more individuals down the hierarchy responding subsequently. This makes R *less* sensitive to α_h compared to any α_k with $k > h$. This allows a higher ranked member of the hierarchy to extract a larger share. When $\rho = 1$, equations (5) and (6) imply that $\xi_h = 0$ for all h for all values of $\alpha > 0$, i.e., resource supply does not respond to resource shares announced by individuals. Individual 1 would choose as high an α_1 as possible. However, R at $\alpha_1 = 1$ drops discontinuously, making individual 1's best response non-existent. Same is true when $\rho > 1$. Hence, $\rho \in (0,1)$ is necessary and sufficient for existence of equilibrium. I therefore assume it throughout the paper.

The resource allocation in a hierarchy is inefficient, due to its unequal distribution. Therefore, presence of a utilitarian donor is not sufficient to ensure efficiency of allocation. However, in order for the model to be more insightful, we need to know what shapes the degree of inefficiency in allocation. This may help us understand conditions under which hierarchies would generate high or low inequality. A hierarchy is characterized by its population, order of the hierarchy and preference. I examine how each of these parameters shapes resource distribution. I then examine how competition in supply and donor's objective affect equilibrium outcome.

For the result on the effect of ρ, we need an additional assumption. Notice that for any CRRA function, $\lim_{\rho\to 0}\pi^{'}(x) = f$ for any given $x > 0$, where f is a positive constant.

Assumption 1 $f > c$.

Proposition 2 *Suppse Assumption 1 holds. Then* $\lim_{\rho\longrightarrow 0}\alpha^* = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$ *and* $\lim_{\rho\longrightarrow 0}\frac{R^*(\alpha^*)}{N} = \bar{R}$.

Proof: I first make three observations:

Lemma 2 β_h^* *is increasing in* ρ *for all* $h \leq H$.

Proof: Fix any $(1-a_h)$. ξ_h is increasing β_h at β_h^*. A higher ρ makes $\xi_h(\alpha^*) < 1$. Hence to restore optimality of β_h, it must increase. □

Lemma 3 $\lim_{\rho\longrightarrow 0}\alpha^* > 0$.

Proof: We can write α_{H+1}^* as

$$\alpha_{H+1}^* = \frac{1}{N-H}\prod_{h=1}^{H}(1-\beta_h^*)$$

Therefore, α_{H+1}^* is decreasing in ρ because of Lemma 2. Hence, $\lim_{\rho\longrightarrow 0}\alpha_{H+1}^* > 0$, which implies $\lim_{\rho\longrightarrow 0}\alpha^* > 0$ because of Proposition 1. □

Lemma 4 *For any* $\rho > 0$, *at* $\alpha = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$, $\pi^{'}(\alpha_h R^*(\alpha)) = c$ *and* $\Delta_h = 0$ *for all* h.

Proof: Follows directly from the equations (4), (5) and (6). □

Now, equation (6) and Lemma 3 imply that $\lim_{\rho\longrightarrow 0}\Delta_H^* = 0$. Given this, equation (5) implies $\lim_{\rho\longrightarrow 0}\Delta_h^* = 0$ for all h. Therefore, given Lemma 4, we get that α^* must approach $(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$ in the limit. To see this notice that if α^* does not approach $(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$, then α_{H+1}^* would remain strictly below $\frac{1}{N}$ in the limit, and hence $\pi^{'}(\alpha_{H+1}^* R^*(\alpha^*))$ would remain strictly above c in the limit (under Assumption 1). We can rewrite equation (4) as

$$\sum \alpha_h^* \Delta_h^* = \pi^{'}(\alpha_{H+1}^* R^*(\alpha^*)) - c$$

If the RHS remains positive in the limit, some Δ_h^* must remain positive in the limit, which is not possible.

The second result follows from the observations that $R^*(\alpha)$ is continuous for $\alpha > 0$ and $\frac{1}{N}R^*((\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})) = \bar{R}$. ■

The proof above shows that a smaller ρ, i.e., making the payoff functions less concave, somewhat surprisingly, results in the first ranked individual extracting *smaller* share of resources.[8] This happens because, smaller ρ makes marginal benefit of supplying resource less sensitive to R. Therefore, in response to changes in α_1, optimization by the donor involves larger changes in R, i.e., $R(\alpha)$ becomes *more* elastic to α_1. This limits the first ranked individual from extracting a higher share. When individual payoffs are approximately risk-neutral, donor's marginal benefit of R becomes almost insensitive to changes in R. Consequently, the elasticity of $R(\alpha)$ w.r.t. α_1 becomes arbitrarily large, almost eliminating the first ranked individuals' power to extract larger shares. In the limit α_1 therefore approaches $1/N$. Since $\alpha_h > \alpha_{h+1}$ and $\sum \alpha_h = 1$, resource distribution becomes approximately equal in the limit. Aggregate resource allocation therefore achieves approximate efficiency in this case. Notice, however, when payoffs are exactly linear, the best strategy of individual ranked 1 is to choose $\alpha_1 = 1$. This is because when payoffs are linear, the donor's resource supply becomes insensitive to extraction strategies of individuals. Therefore, the limit equilibrium and the equilibrium *at the limit* are different.

Lemma 5 *The following statements are true:*

(i) $R^*(\alpha)$ *is increasing in* N *for any* $\alpha > 0$.

(ii) $R^*(\alpha)/N - H$ *is decreasing in* N *for any* $\alpha > 0$.

(iii) For all $h \leq H$, β_h^* *is decreasing in* N.

(iv) $R^*(\alpha^*) \to \infty$ *as* $N \to \infty$.

Proof: Fix any $\alpha > 0$. (i) $R^*(\alpha)$ must satisfy equation (4) for all values of N $(> H)$. Increasing N would increase the LHS of equation (4) if $R^*(\alpha)$ stays the same or falls. Therefore, $R^*(\alpha)$ must increase. (ii) If $R^*(\alpha)/N - H$ stays the same or increases, then the LHS would fall with a higher N. Hence $R^*(\alpha)/N - H$ must decrease with N.

(iii) Fix some $(1 - a_{h-1}) > 0$. Examining equations (5) and (6), we get that increasing N would make $\xi_h(\alpha^*) > 1$ if β_h^* stays the same (since $R^*(\alpha)$ rises and $R^*(\alpha)/N - H$ falls). At β_h^*, ξ_h is increasing in β_h. Therefore, to restore equilibrium β_h^* must fall.

(iv) Finally, as $N \to \infty$, we must have $\pi'(\alpha_{H+1}^* R^*(\alpha)) \to 0$ (from equation (4)). But α_{H+1}^* must approach zero in the limit. Hence, $R^*(\alpha^*) \to \infty$. ■

[8] For individuals ranked 2 and below, β_h^* falls as ρ falls, i.e., they extract a smaller fraction of the residual share (Lemma 2), but α_h^* need not necessarily fall. See Figure 1b and its discussion in Section III.II.

III.II Large Hierarchies

I now examine the property of the equilibrium resource distribution when population becomes large, for any fixed H. I refer to them as *large population hierarchies* and denote them by $\langle H,\pi,U\rangle$. Large population hierarchies where every individual has a different rank has an infinite order, and are denoted by $\langle \pi,U\rangle$; they are referred to as *large hierarchies*.

Proposition 3 *The equilibrium shares in a large population hierarchy of order H are given by*

$$\lim_{N\to\infty}\alpha_h^* \;=\; \rho(1-\rho)^{h-1} \quad \textit{for all } h\leq H$$

Proof: It is sufficient to prove that $\lim_{N\to\infty}\beta_h^* = \rho$ for all h. The optimality of $R^*(\alpha^*)$ is given by

$$\begin{aligned}
\sum_{h=1}^{H}\alpha_h^*\pi^{'}(\alpha_h^*R^*(\alpha^*)) + (N-H)\alpha_{H+1}^*\pi^{'}\left(\alpha_{H+1}^*R^*(\alpha^*)\right) &= c\\
\Rightarrow\quad (1-a_{H-1}^*)\pi^{'}\left(\alpha_{H+1}^*R^*(\alpha^*)\right)+\sum_{h=1}^{H-1}\alpha_h^*\pi^{'}(\alpha_h^*R^*(\alpha^*)) &= \alpha_H\Delta_H^*+c = \frac{c}{1-\rho}\\
\Rightarrow\quad (1-a_{H-1}^*)\pi^{'}\left(\alpha_{H+1}^*R^*(\alpha^*)\right) &= \frac{c}{1-\rho}-\sum_{h=1}^{H-1}\alpha_h^*\pi^{'}(\alpha_h^*R^*(\alpha^*))
\end{aligned} \tag{11}$$

Since β_h^* are all monotone in N (Lemma 5 (iii)) and are bounded, $\lim_{N\to\infty}\beta_h^*$ exists for all $h\leq H$. Hence, $\lim_{N\to\infty}\alpha_h^*$ exists for all $h\leq H+1$. Let's denote $\lim_{N\to\infty}\alpha_h^*\equiv\bar{\alpha}_h^*$ (and $\lim_{N\to\infty}a_h^*\equiv\bar{a}_h^*$). Taking limit on both sides of the equation (6) we get,

$$\bar{\alpha}_H^*\left[\lim_{N\to\infty}\pi^{'}\left(\alpha_{H+1}^*R^*(\alpha^*)\right)-\lim_{N\to\infty}\pi^{'}(\alpha_H^*R^*(\alpha^*))\right] \;=\; \frac{\rho c}{1-\rho}$$

Now $R^*(\alpha^*)\to\infty$ in the limit (Lemma 5 (iv)). Also, $\bar{\alpha}_H^*>0$, since $\bar{\alpha}_H^*=0$ would imply $\bar{\alpha}_{H+1}^*=0$ and hence, $\Delta_H^*<\infty$ and $\xi_H^*=0$ in the limit, contradicting equation (6). Hence, $\pi^{'}(\alpha_H^*R^*(\alpha^*))\to 0$. Moreover, from equation (11),

$$(1-\bar{a}_{H-1}^*)\lim_{N\to\infty}\pi^{'}\left(\alpha_{H+1}^*R^*(\alpha^*)\right) \;=\; \frac{c}{1-\rho}$$

provided $\bar{\alpha}_h^*>0$ for all h, which is true given $\bar{\alpha}_H^*>0$ and Proposition 1. Therefore,

$$\bar{\alpha}_H^*\frac{c}{1-\rho}\frac{1}{(1-\bar{a}_{H-1}^*)} \;=\; \frac{\rho c}{1-\rho}$$

$$\Rightarrow\quad \bar{\alpha}_H^* \;=\; \rho(1-\bar{a}_{H-1}^*)$$

The equation above also holds for any fixed value of a_{H-1}. We then get

$$\lim_{N\to\infty}\frac{d\alpha_H^*}{d\alpha_{H-1}}=\frac{d\bar{\alpha}_H^*}{d\alpha_{H-1}} \;=\; -\rho$$

The equilibrium condition of α_{H-1} gives us,

$$\alpha_{H-1}^*\left[\pi'\left(\alpha_{H+1}^*R^*(\alpha^*)\right)\left\{1+\frac{d\alpha_H^*}{d\alpha_{H-1}^*}\right\}-\frac{d\alpha_H^*}{d\alpha_{H-1}^*}\pi'(\alpha_{H-1}^*R(\alpha^*))-\pi'(\alpha_H^*R^*(\alpha^*))\right] \;=\; \frac{\rho c}{1-\rho}$$

Taking limit on both sides gives us,

$$\begin{aligned}
\bar{\alpha}_{H-1}^*\frac{c}{1-\rho}\frac{1}{(1-\bar{a}_{H-1}^*)}\{1-\rho\} &= \frac{\rho c}{1-\rho}\\
\Rightarrow\quad \bar{\alpha}_{H-1}^*\frac{c}{1-\rho}\frac{1}{\{(1-\bar{a}_{H-2}^*)-\bar{\alpha}_{H-1}^*\}}\{1-\rho\} &= \frac{\rho c}{1-\rho}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad \bar{\alpha}_{H-1}^* &= \frac{\rho}{1-\rho}\{(1-\bar{a}_{H-2}^*)-\bar{\alpha}_{H-1}^*\}\\
\Rightarrow\quad \bar{\alpha}_{H-1}^* &= \rho(1-\bar{a}_{H-2}^*)
\end{aligned}$$

We now use the method of induction to prove that $\bar{\alpha}_{H-k}^*=\rho(1-\bar{a}_{H-k-1}^*)$ for all $k=0,1,\dots,H-1$. The statement has been proven to be true for $k=0$ and $k=1$. Suppose that the statement is true for all h up to some $k-1<H-1$. We now show that the statement is true for $h=k$. First,

$$\begin{aligned}
\bar{\alpha}_H^* \;=\; \rho(1-\bar{a}_{H-1}^*) \;=\; \rho(1-\bar{a}_{H-2}^*-\rho(1-\bar{a}_{H-2}^*)) &= \rho(1-\rho)(1-\bar{a}_{H-2}^*)\\
&= \rho(1-\rho)^2(1-\bar{a}_{H-3}^*)\\
&= \dots = \rho(1-\rho)^{k-1}(1-\bar{a}_{H-k}^*)
\end{aligned}$$

Following the same logic we can show that, generally,

$$\bar{\alpha}_{H-l}^* \;=\; \rho(1-\rho)^{k-l-1}(1-\bar{a}_{H-k}^*) \quad \text{for all } l=0,1,\dots,k-1$$

Hence, for all $l=0,1,\dots,k-1$.

$$\frac{d\bar{\alpha}_{H-l}^*}{d\bar{\alpha}_{H-k}^*} \;=\; -\rho(1-\rho)^{k-l-1}$$

$$\Rightarrow\quad \lim_{N\to\infty}\frac{d\alpha_{H-l}^*}{d\alpha_{H-k}^*} \;=\; -\rho(1-\rho)^{k-l-1}$$

$$\Rightarrow \quad \lim_{N\to\infty}\left[1+\sum_{h=H-k+1}^{H}\frac{d\alpha^*_{H-l}}{d\alpha^*_{H-k}}\right] \;=\; 1-\sum_{l=0}^{k-1}\rho(1-\rho)^{k-l-1}$$
$$= \; 1-\rho-\rho(1-\rho)-\ldots-\rho(1-\rho)^{k-1}=(1-\rho)^k$$

Now, the optimality condition of α^*_{H-k} is given by

$$\alpha^*_{H-k}\left[\pi'\left(\alpha^*_{H+1}R^*(\alpha^*)\right)\left\{1+\sum_{h=H-k+1}^{H}\frac{d\alpha^*_{H-l}}{d\alpha^*_{H-k}}\right\} - \right.$$
$$\left.\sum_{h=H-k+1}^{H}\frac{d\alpha^*_{H-l}}{d\alpha^*_{H-k}}\pi'(\alpha^*_hR^*(\alpha^*))-\pi'(\alpha^*_{H-k}R^*(\alpha^*))\right] \;=\; \frac{\rho c}{1-\rho}$$

Taking limit on both sides we get,

$$\bar{\alpha}^*_{H-k}\frac{c}{1-\rho}\frac{1}{(1-\bar{a}^*_{H-1})}\{(1-\rho)^k\} \;=\; \frac{\rho c}{1-\rho}$$

Notice that $(1-\bar{a}^*_{H-1}) \;=\; (1-\rho)(1-\bar{a}^*_{H-2}) \;=\; \ldots \;=\; (1-\rho)^k(1-\bar{a}^*_{H-k-1})$. Hence,

$$\bar{\alpha}^*_{H-k} \;=\; \rho(1-\bar{a}^*_{H-k-1})$$

Hence the statement is true for k. ∎

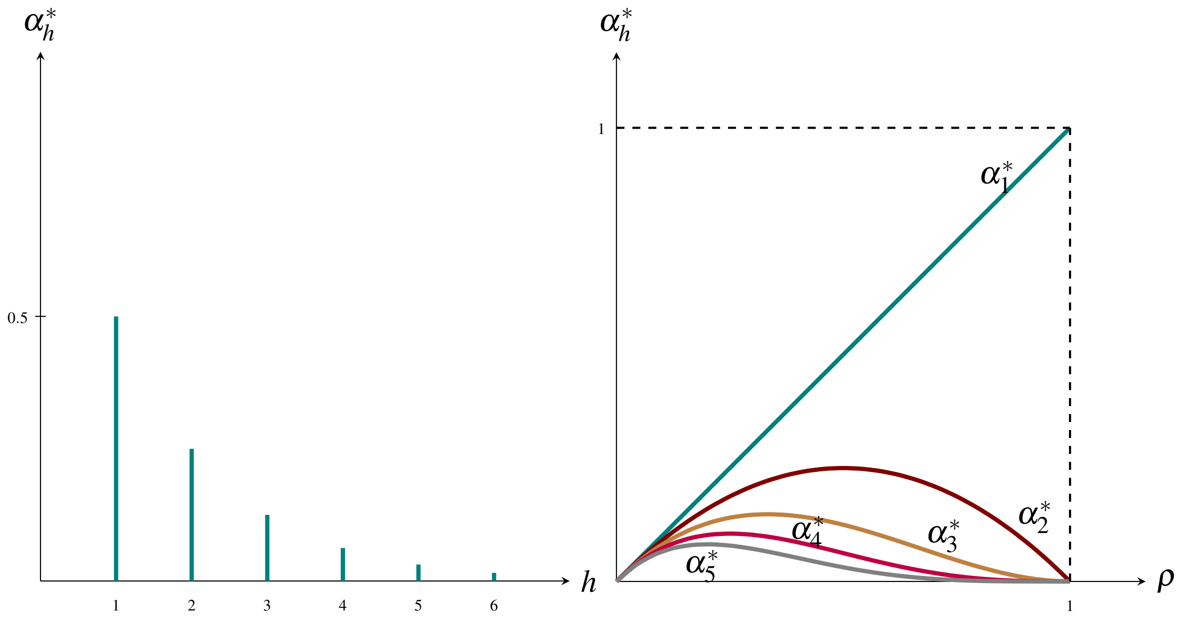
Corollary 1 *The equilibrium shares in a large hierarchy are given by*

$$\bar{\alpha}_h \;=\; \rho(1-\rho)^{h-1} \quad \textit{for all } h\geq 1$$

The limiting resource distribution in large hierarchies therefore follows a geometric distribution over the ranks with parameter ρ. As the proof of the result highlights, in equilibrium, individuals have identical strategies in large hierarchies given by $\alpha_h=\rho(1-a_{h-1})$ where $a_h=\sum_{k=1}^{h}\alpha_k$. Hence, each individual extracts ρ fraction of the residual share left after higher ranked individual choose their shares.

We can think about an alternate interpretation of the equilibrium distribution. Suppose the donor chooses to allocate some discrete prize using the following lottery. He flips a coin with probability of head being ρ. The donor stops when the first head appears. If the first head appears in h^{th} flip then, the h^{th} individual receives the prize. The probability of rank holder h receiving the prize would then be $\rho(1-\rho)^{h-1}$. This gives us an understanding of the nature of advantage higher rank holders get in hierarchies.

Figure 1a depicts the distribution of α^*_h for the first 6 ranks when $\rho=0.5$. Figure 1b plots α^*_h as a function of ρ for the first five ranks. While α_1 falls monotonically as ρ becomes smaller, all the other shares are non-monotonic in ρ. α^*_2, for example, attains its peak for $\rho=\frac{1}{2}$, α^*_3 for $\rho=\frac{1}{3}$ and so on. The ratio of α^*_h and α^*_{h-1} is $(1-\rho)$, which increases monotonically



(a) Distribution of α_h^* for $\rho = 0.5$

(b) α_h^* as function of ρ

Figure 1—α_h^* for different ranks h and for different values of ρ

as ρ becomes smaller. Therefore, the shares in equilibrium become more equal as the payoff concavity decreases. In the limit, all shares become equal, as predicted by Proposition 2.

Efficiency and Welfare in Large Population Hierarchies: Let $\bar{R}^*(H)$ denote the equilibrium per capita resource allocation in large population hierarchies of order H. Then $\bar{R}^*(0) = \bar{R}$, the efficient par capita resource allocation defined above. Let's denote the welfare or donor's payoff in a large population hierarchy as $U^*(\rho,H)$. Again, $U^*(\rho,0) = \bar{U}$, the welfare level in the benchmark case. The following result establishes that hierarchies are inefficient (and welfare reducing), and inefficiency and welfare loss increase with the order of hierarchy.

Lemma 6 *$\bar{R}^*(H)$ and $U^*(\rho,H)$ are decreasing in H for all $H \geq 0$.*

Proof: Given Proposition 3, in large hierarchies, $\alpha_r^* = 1 - \sum_{h=1}^{H} \alpha_h^* = (1-\rho)^H$. Moreover, equation (4) in large hierarchies gives

$$\alpha_r^* \pi'(\alpha_r^* \bar{R}^*(H)) = c$$

$$\Rightarrow R^*(H) = \frac{1}{(1-\rho)^H} \pi'^{-1}\left(\frac{c}{(1-\rho)^H}\right)$$

Treating H as a continuous variable and differentiating $\bar{R}^*(H)$ w.r.t. H gives

$$\frac{d\bar{R}^*(H)}{dH} = \bar{R}^*(H)\frac{1-\rho}{\rho}\ln(1-\rho) < 0 \text{ for all } H \geq 0$$

Moreover, $U^*(\rho,H)$ is given by,

$$\begin{aligned} U^*(\rho,H) &= \pi\left(\pi'^{-1}\left(\frac{c}{(1-\rho)^H}\right)\right) - \frac{c}{(1-\rho)^H}\pi'^{-1}\left(\frac{c}{(1-\rho)^H}\right) \\ &= \int_0^{\pi'^{-1}\left(\frac{c}{(1-\rho)^H}\right)} \left[\pi'(z) - \frac{c}{(1-\rho)^H}\right] dz \end{aligned}$$

$U^*(\rho,H)$ is decreasing in H for all $H \geq 0$ since for any positive integer k,

$$\pi'^{-1}\left(\frac{c}{(1-\rho)^H}\right) > \pi'^{-1}\left(\frac{c}{(1-\rho)^{H+k}}\right) \quad \text{and} \quad \left[\pi'(z) - \frac{c}{(1-\rho)^H}\right] > \left[\pi'(z) - \frac{c}{(1-\rho)^{H+k}}\right]$$

■

IV Resource "Market" in Hierarchies

In this section I consider the case of multiple donors supplying the same resource simultaneously, i.e., there is a resource "market" in which members of hierarchy can engage in extraction. Examples include the aid (or research funding) market with multiple aid (or research grant) agencies, multiple regional governments allocating contracts for public projects etc. Suppose that there are $J \geq 1$ identical donors, indexed by $j = 1,2,\ldots,J$ in the resource market for R. The marginal cost of resource supply for each donor is c. Each rank holding individual now announces a vector of shares $\tilde{\alpha}_h = (\alpha_h^1, \alpha_h^2, \ldots, \alpha_h^J)$. An element α_h^j in the vector corresponds to the share of resource that the individual with rank h commits to extract from the resource supplied by donor j. The individual, by announcing $\tilde{\alpha}_h$, therefore, makes the commitments simultaneously across all donors. As before, the sequence of commitments is given by the ranks of the individuals in the hierarchy. $\alpha_h^j \in [0, 1 - a_{h-1}^j]$ where $a_0^j = 0$ for all j and $a_h^j = \sum_{k=1}^h \alpha_k^j$ for $h \geq 1$ for all j. All the J donors observe $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_H, \tilde{\alpha}_{H+1})$ and simultaneously choose their respective resource supply R^j. Let $\tilde{R} = (R^1, R^2, \ldots, R^J)$. Each donor takes the resource supplied by other donors as given and chooses its supply to maximize the following payoff:

$$U(R^j, \tilde{R}^{-j}) = \frac{1}{N}\left\{\sum_{h=1}^{H} \pi(\alpha_h^j R^j + \tilde{\alpha}_h^{-j} \cdot \tilde{R}^{-j}) + (N-H)\pi(\alpha_{H+1}^j R^j + \tilde{\alpha}_{H+1}^{-j} \cdot \tilde{R}^{-j}) - cR^j\right\}$$

where $\tilde{R}^{-j}$ and $\tilde{\alpha}_h^{-j}$ are the vectors $\tilde{R}$ and $\tilde{\alpha}_h$, respectively, without their respective j^{th} elements. $\tilde{\alpha}_h \cdot \tilde{R}$ is the dot product of the vectors $\tilde{\alpha}_h$ and $\tilde{R}$. The optimal profile of resource supply strategies $\tilde{R}^*(\tilde{\alpha}) = (R^{*1}(\tilde{\alpha}), R^{*2}(\tilde{\alpha}), \ldots, R^{*J}(\tilde{\alpha}))$ constitute a Nash Equilibrium given $\tilde{\alpha}$, if

$$\sum_{h=1}^{H} \alpha_h^j \pi'(\tilde{\alpha}_h \cdot \tilde{R}^*(\tilde{\alpha})) + (N-H)\alpha_{H+1}^j \pi'\left(\tilde{\alpha}_{H+1} \cdot \tilde{R}^*(\tilde{\alpha})\right) = c \quad \text{for all } j$$

I focus on symmetric strategy equilibria of the game. Since all donors are identical and all members of hierarchy have identical payoffs, symmetric strategies are natural candidates for equilibrium. Let individual h's symmetric strategy be denoted by $\alpha_h^j = \alpha_h \in [0,1]$ for all j. Then, we can represent the choices of all members of hierarchy by a vector of shares $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_H, \alpha_{H+1})$. The following result follows:

Proposition 4 *For any $J > 1$, a symmetric equilibrium $(\alpha^*, \tilde{R}^*(\alpha))$ exists and is unique. Moreover,*

(i) $R^{**}(\alpha) = \sum_j R^{*j}(\alpha)$ *is independent of J,*

(ii) $\alpha^* > 0$ *and* $\alpha_h^* > \alpha_{h+1}^*$ *for all* $h \leq H$,

(iii) for any fixed N, $\lim_{J\to\infty} \alpha_h^* = \frac{1}{N}$, *and*

(iv) for any fixed J, $\lim_{N\to\infty} \alpha_h^* = \frac{\rho}{J}\left(1 - \frac{\rho}{J}\right)^{h-1}$

Proof: See Appendix Section A.I. ■

In the limit equilibrium when the number of donors becomes large, distribution of resource becomes approximately efficient for any ρ. However, the source of this positive result is distinct from the standard result in the oligopoly case where we get efficiency as the number of firms becomes infinitely large. In the oligopoly case, the source of inefficiency is in the supply side of the market. Hence, increasing competition among firms increases equilibrium supply. In this model, the source of the inefficiency is in the demand side, due to the presence of hierarchy. Moreover, the aggregate supply function $R^*(\alpha)$ remains fixed for all J. Hence, we get efficiency in demand *without* affecting aggregate supply function.[9] Competition in the supply side, therefore, disciplines the demand side of the market. This happens because, when there are multiple donors, choosing to extract higher share from a given donor leads to reduced supply from other donors as well. This is because the marginal benefit of supply by one donor depends on the resource distribution of other donors' supply. Hence the aggregate resource becomes more elastic with respect to the extraction strategy of individuals w.r.t. any donor. This limits the power of those ranked higher to extract more, and consequently, reduces inequality in resource distribution and improves efficiency in allocation.

V Hierarchies with an Egalitarian Donor

In certain contexts the donor may not behave as a utilitarian. Consider, for example, a foreign aid agency whose reputation depends on the amount of aid received by the citizens, located at the bottom of the hierarchy. In such a context, the donor would behave as an egalitarian, i.e., maximizing the Rawlsian social welfare. In this section, I examine the consequence of having a

[9]Equilibrium aggregate supply, $R^*(\alpha^*)$, does increase with J and reaches the efficient level in the limit.

Rawlsian donor on equilibrium resource distribution. $\pi(\cdot)$, as before, is CRRA with $\rho \in (0,1)$. As before, individuals in a hierarchy with a Rawlsian donor sequentially commits to α_h. The donor observes the vector of commitments α and chooses $R^e(\alpha)$ where

$$R^e(\alpha) \;=\; \operatorname*{argmax}_R E(R) \equiv \operatorname*{argmax}_R \frac{1}{N}\left\{N \min_{1\leq h\leq H+1}\{\pi(\alpha_h R)\} \;-\; cR\right\} \tag{12}$$

$E(R)$ is the objective of the Rawlsian donor. The Rawlsian donor therefore behaves as if everyone in the hierarchy receives the minimum payoff and maximizes aggregate payoff net of cost with that supposition. Defining $E(R)$ this way makes it is comparable to $U(R)$, the utilitarian objective. To see this, notice that when $\alpha = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$, the optimal per capita R given $E(\cdot)$ is $\bar{R} = \pi'^{-1}(c)$ which is the same as in the utilitarian case. Moreover, the donor's payoff in both cases are also the same. Hence efficient allocation and welfare are identical under the two types of donor. Let $\langle N, H, \pi, E\rangle$ now denote a hierarchy with an egalitarian donor. Writing this way also facilitates comparison of donor objectives in large population hierarchies.[10]

V.I Simple Case: $N = 2$

To fix ideas, let's analyze the case with $N = 2$. Here α is a scalar; it is the commitment made by individual 1. For any choice of $\alpha \in [0,1]$, the optimal resource supply by donor, denoted by $R^e(\alpha)$, is given as

$$R^e(\alpha) \;=\; \operatorname*{argmax}_R \left\{\min\{\pi(\alpha R), \pi((1-\alpha)R)\} - c\frac{R}{2}\right\}$$

Individual 1 chooses α given $R^e(\alpha)$ above. Let α^e be his optimal choice.

Lemma 7 *Equilibrium $(\alpha^e, R^e(\alpha))$ exists and is unique. Moreover, α^e is given by*

$$\alpha^e = \begin{cases} \frac{1}{2} & \textit{if } \rho \in (0, \frac{1}{2}] \\[2ex] \rho & \textit{if } \rho \in (\frac{1}{2}, 1) \end{cases}$$

Proof: Notice that $R^e(\alpha) = 0$ when $\alpha = 0$ or 1. For $\alpha \in (0,1)$, donor's optimization gives us,

$$\alpha\pi'(\alpha R^e(\alpha)) = \frac{c}{2} \quad \text{for } \alpha \in (0, \tfrac{1}{2}) \text{ and,} \tag{13}$$

$$(1-\alpha)\pi'((1-\alpha)R^e(\alpha)) = \frac{c}{2} \quad \text{for } \alpha \in [\tfrac{1}{2}, 1). \tag{14}$$

Given that $\pi'(\cdot)$ is monotonic and satisfies Inada conditions (since it is CRRA), $R^e(\alpha)$ exists and is unique for any $\alpha \in (0,1)$.

[10]Equilibrium shares remain identical if we define $E(R) = \min_{1\leq h\leq H+1}\{\pi(\alpha_h R)\} \;-\; cR$.

At $\alpha = \frac{1}{2}$,

$$\frac{1}{2}\pi'\left(\frac{1}{2}R^e\left(\frac{1}{2}\right)\right) = \frac{c}{2} \tag{15}$$

Comparing equations (13) and (15) we get, for any $\alpha \in (0, \frac{1}{2})$,

$$\pi'(\alpha R^e(\alpha)) > \pi'\left(\frac{1}{2}R^e\left(\frac{1}{2}\right)\right) \quad \Rightarrow \quad \alpha R^e(\alpha) < \frac{1}{2}R^e\left(\frac{1}{2}\right)$$

Therefore, individual 1 never chooses α less than $\frac{1}{2}$. Differentiating equation (14) w.r.t. α gives the elasticity of $R^e(\alpha)$ as

$$\xi = -\frac{dR^e(\alpha)}{d\alpha}\frac{\alpha}{R^e(\alpha)} = \frac{1-\rho}{\rho}\frac{\alpha}{1-\alpha} \tag{16}$$

ξ is positive and increasing in α. Individual 1 chooses $\alpha \in [\frac{1}{2}, 1)$ such that $\xi = 1$ or is as close to 1 as possible. This yields $\alpha^e = \min\{\rho, \frac{1}{2}\}$. ■

In the simple case, the equilibrium allocation under an egalitarian donor is therefore starkly different from the one under a utilitarian donor. Under egalitarian donor, allocation is efficient for half of the parameter range. Moreover, it is easy to show that for all $\rho \in (0,1)$, $\alpha^e < \alpha^*$. This happens because egalitarian donor's marginal benefit of resource is less sensitive to changes in α, making $R(\alpha)$ more elastic. This makes the equilibrium shares less unequal than the utilitarian case.

V.II General Case

Before characterizing the equilibrium for the general case, I first examine the equilibrium when the donor's payoff depends on only the lowest ranked individual:

Lemma 8 *Suppose $\hat{E}(R) = \frac{1}{N}\{N\pi(\alpha_{H+1}R) - cR\}$. Then equilibrium $(\alpha^e, R^e(\alpha))$ exists and is unique. Moreover, for all $h \leq H$,*

$$\alpha_h^e = \rho(1-\rho)^{h-1}$$

Proof: See Appendix Section A.II. ■

If the donor's objective depends only on the payoff of the lowest ranked individual, the equilibrium shares in any hierarchy follow the Geometric distribution, i.e., they are same as that under utilitarian donor in large population hierarchies. This happens because donor objective $U(\cdot)$ approaches $\hat{E}(\cdot)$ as N becomes large. In fact, consider any vector of welfare weights $\omega = (\omega_1, \omega_2, \ldots, \omega_{H+1}) > 0$ such that $\sum_{h=1}^{H+1} \omega_h = 1$ with

$$\tilde{U}(R) = \frac{1}{N}\left\{\sum_{h=1}^{H} \omega_h \pi(\alpha_h R) + (N-H)\omega_{H+1}\pi(\alpha_{H+1}R) - cR\right\}$$

as the generalized utilitarian objective of a donor. Then, the equilibrium share would approach the Geometric distribution in large population hierarchies, since $\tilde{U}(R) \longrightarrow \hat{E}(R)$ when $N \longrightarrow \infty$.

Proposition 5 *In a hierarchy $\langle N, H, \pi, E\rangle$, equilibrium $(\alpha^e, R^e(\alpha))$ exists and is unique. α_h^e, for all $h \leq H+1$, is given by*

$$\begin{aligned} \alpha_h^e &= \frac{1}{N-h+1}(1-a_{h-1}^e) \quad \text{if } \rho \in (0, \tfrac{1}{N-h+1}] \\ &= \rho(1-a_{h-1}^e) \qquad\qquad \text{if } \rho \in (\tfrac{1}{N-h+1}, 1) \end{aligned}$$

Proof: See Appendix Section A.III. ■

The Proposition implies that in hierarchies with an egalitarian donor, individuals either chooses the efficient β_h or $\beta_h = \rho$. The equilibrium vector of resource shares is then given by,

$$\begin{aligned} \alpha^e &= \left(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right) \quad \text{if } \rho \in (0, \tfrac{1}{N}] \\ &= \left(\rho, \frac{1-\rho}{N-1}, \ldots, \frac{1-\rho}{N-1}\right) \quad \text{if } \rho \in (\tfrac{1}{N}, \tfrac{1}{N-1}] \\ &\vdots \\ &= \left(\underbrace{\rho, \rho(1-\rho), \ldots, \rho(1-\rho)^{k-1}}_{k \text{ individuals}}, \underbrace{\frac{(1-\rho)^k}{N-k}, \ldots, \frac{(1-\rho)^k}{N-k}}_{N-k \text{ individuals}}\right) \quad \text{if } \rho \in (\tfrac{1}{N-k+1}, \tfrac{1}{N-k}] \\ &\vdots \\ &= \left(\rho, \rho(1-\rho), \ldots, \rho(1-\rho)^{k-1}, \ldots, \rho(1-\rho)^{H-1}, \frac{(1-\rho)^H}{N-H}\right) \quad \text{if } \rho \in (\tfrac{1}{N-H+1}, 1) \end{aligned}$$

The allocation is efficient for $\rho \in (0, 1/N]$. Equilibrium shares follow Geometric distribution for $\rho \in (1/N - H + 1, 1)$. For intermediate ranges of ρ, a subset of individuals starting from the top chooses $\beta_h = \rho$, while the rest get equal shares. However, for any given H, as N becomes large, α_h^e approaches $\rho(1-\rho)^{h-1}$ for all values of ρ. Therefore, large population hierarchies with an egalitarian donor exhibit the same equilibrium as with any generalized utilitarian donor.

Comparison across Donor Objectives: Notice that with a utilitarian donor, $\beta_h^* > \rho$ for all $h \leq H$ in any finite hierarchy, since β_h^* is decreasing in N and equals ρ in the limit. Therefore, $\beta_h^* > \beta_h^e$, i.e., all individuals extract a greater fraction of the residual share when the donor is utilitarian. Hence, $\alpha_{H+1}^* < \alpha_{H+1}^e$ (from equation (3)) – residual claimants in a hierarchy are better off under an egalitarian donor. For the top ranked individual, $\alpha_1^* > \rho$ and $\alpha_1^* > 1/N$ in any finite hierarchy. Hence $\alpha_1^* > \alpha_1^e$ – the top ranked individual is worse off under egalitarian donor. For the intermediately ranked individuals, the comparison would depend on ρ.

Generalized Egalitarian Objective: Let's define a *generalized* egalitarian objective of a donor in the following way: take any vector of welfare weights $\omega = (\omega_1, \omega_2, \dots, \omega_{H+1}) \geq 0$ such that $\omega_{H+1} > 0$ and $\sum_{j=1}^{H+1} \omega_j = 1$. Define $o(\alpha) = (o_1(\alpha), o_2(\alpha), \dots, o_{H+1}(\alpha))$ as the order statistic mapping; $o_h(\alpha)$ gives the position of α_h in α when individual shares are ordered in descending manner. Therefore, $o_h(\alpha) = 1$ implies that α_h is the highest share in α. Then, donor's objective

$$\tilde{E}(R) = \frac{1}{N}\left\{\sum_{h=1}^{H} \omega_{o_h(\alpha)}\pi(\alpha_h R) + (N-H)\omega_{o_{H+1}(\alpha)}\pi(\alpha_{H+1}R) - cR\right\}$$

is a representation of a generalized egalitarian objective. The donor has welfare weight ω_h on the individual with the h^{th} highest value of α_h in any α, i.e., welfare weights are endogenous. The Rawlsian objective is a special case of this when $\omega_{H+1} = 1$. Notice that for all α such that $\alpha_h > \alpha_{h+1}$, we have $o_h(\alpha) = h$. Therefore, for that set of α, welfare weights are constant; individual h has weight ω_h. Hence, the equilibrium $(\alpha^*, R^*(\alpha))$ with a generalized utilitarian donor with welfare weights ω is also the equilibrium in the generalized egalitarian donor case with the same welfare weights, since in equilibrium we have $\alpha_h^* > \alpha_{h+1}^*$. Moreover, notice that $\tilde{E}(R) \longrightarrow E(R)$ as $N \longrightarrow \infty$, i.e., any generalized egalitarian welfare function approaches the egalitarian welfare function, for N large. Therefore, the equilibrium shares in the large population hierarchy with generalized egalitarian donor will also follow the Geometric distribution. Hence, for a large class of donor objectives, the large population hierarchies, and consequently, large hierarchies exhibit identical inequality and welfare.

VI Discussion

Policy Implication: In many organizations the higher ranked individuals, through laws and regulations, have externally imposed limits on their consumption. The head of state, for example, is subject to various executive constraints that attempt to limit their rent extraction. Similarly, in the context of firms, scholars have argued that there should be limits on managerial or executive compensations to curb within firm inequality in pay. Let's consider such a policy in the light of this model. Suppose that the first ranked individual can only extract $\bar{\alpha}_1 < \rho$ share of the resource. Then in large population hierarchies, the equilibrium share of other individuals are given by $\alpha_h^* = \rho(1-\rho)^{h-2}(1-\bar{\alpha}_1) > \rho(1-\rho)^{h-1}$. Hence, other individuals would extract more. In large hierarchies, therefore, constraints on the executive (i.e., highest ranked individual) may not result in significant fall in inequality as it would induce others to extract larger shares. If the head of the state is constrained to gain financially from the position, for example, her family members, business partners, friends etc., who could be thought of as ranked immediately below, may begin to extract higher rent. More generally, limiting the extraction of some top ranked individuals does not improve inequality among the rest, since the Geometric distribution is memoryless.

Group Based Hierarchies: Often hierarchies exist not across individuals but groups. Caste system in India is an example of group based hierarchy, where several individuals in society can be members of a given caste, which is associated with a given rank in the hierarchy. Hierarchies in organizations are also typically group based, where each tier or rank in the hierarchy is associated with some position in the organization such as worker, manager, executive etc. and several individuals can belong to a single rank. The model can accommodate this by assuming that all individuals in a given rank act jointly to decide the group share and then divide the spoil equally among group members. Therefore, if there are n_h members in rank h, then they jointly commit to extracting α_h share of the resource and then each member receives α_h/n_h share of the resource. When number of individuals in a given rank is same across all ranks, all the results would continue to hold due to symmetry across ranks w.r.t. their size. Moreover, if the size distribution of ranks is pyramidal, i.e., higher ranks are smaller in size, as is often the case in organizations (Garicano 2000), inequality in allocation across individuals with different ranks would *increase*. This is because larger group size disincentivizes individuals to extract more since the group allocation gets divided among a larger population. Therefore, even though α_h is increasing in n_h, α_h/n_h is decreasing in group size. This however does not necessarily imply that social welfare would be lower in group based hierarchies relative to individual based ones. This is because moving to a group based hierarchy would involve some individuals' rank improving which could potentially offset higher inequality across ranks. Moreover, if a social planner wanted to divide n individuals across $H+1$ groups, it would choose larger groups in higher ranked positions, i.e., $n_h \geq n_{h+1}$. Hence, rent extraction concerns may impose some constraints on the organizational structure.

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A Proofs of Results

A.I Proof of Proposition 4

Suppose that the members of hierarchy follows some symmetric strategy given by α. The best response function of donor j given $R^{-j}(\alpha)$ is given by

$$\sum_{h=1}^{H} \alpha_h \pi^{'}\left(\alpha_h R^j(\alpha) + \alpha_h \sum_{k \neq j} R^k(\alpha)\right) + (N-H)\alpha_{H+1}\pi^{'}\left(\alpha_{H+1} R^j(\alpha) + \alpha_{H+1} \sum_{k \neq j} R^k(\alpha)\right) = c$$

Hence, the subgame of donors has a unique symmetric Nash Equilibrium $R^{j*}(\alpha) = R^*(\alpha)$ for all j, where $R^*(\alpha)$ is given by

$$\sum_{h=1}^{H} \alpha_h \pi^{'}\left(\alpha_h J R^*(\alpha)\right) + (N-H)\alpha_{H+1}\pi^{'}\left(\alpha_{H+1} J R^*(\alpha)\right) = c \tag{17}$$

Let's define $R^{**}(\alpha) = JR^*(\alpha)$. The equation above implies that $R^{**}(\alpha)$ is independent of J. This proves (i) of the Proposition. Moreover, $R^{**}(\alpha)$ has the same property as that of the equilibrium supply under a single donor case as the equation above is identical to equation (4).

Each individual h maximizes $\alpha_h R^{**}(\alpha) = J\alpha_h R^*(\alpha)$. The optimality condition for h is then given by

$$\xi_h = \frac{1}{J}$$

where $\xi_h = -\frac{dR^*(\alpha)}{d\alpha_h}\frac{\alpha_h}{R^*(\alpha)}$ is the elasticity of $R^*(\alpha)$ w.r.t. α_h. Deriving the expression for ξ_h from equation (17) we get that equilibrium α^* must satisfy

$$\alpha_H^* \Delta_H^* = \frac{c\rho/J}{1-\rho/J} \tag{18}$$

and for $h < H$,

$$\alpha_h^* \left[\Delta_h^* - \sum_{k=h+1}^{H} \theta_h^{k*} \Delta_k^*\right] = \frac{c\rho/J}{1-\rho/J} \tag{19}$$

These two equations, therefore, are identical to the $J = 1$ case with ρ being replaced by ρ/J. Hence, following the same logic as in the proof of Proposition 1 we get that α^* exists, is unique, $\alpha^* > 0$ and $\alpha_h^* > \alpha_{h+1}^*$. This proves (ii). Proof of (iii) follows the same logical steps of the proof of Proposition 2. Result (iv) follows from equations (18) and (19) and the proof of Proposition 3 with ρ being replaced by ρ/J.

A.II Proof of Lemma 8

Existence and uniqueness of equilibrium follow from arguments similar to the proof of Proposition 1. I prove the last result by induction. I first show that it is true for H and $H-1$. $R^e(\alpha)$

is given by

$$\alpha_{H+1}\pi^{'}(\alpha_{H+1}R^e(\alpha)) = \frac{c}{N} \tag{20}$$

Now, using $\alpha_h = \beta_h(1-a_{h-1})$ we get $\alpha_{H+1} = \frac{1-\beta_H}{N-H}(1-a_{H-1})$. Then for any fixed $(1-a_{H-1})$, differentiating the equation above w.r.t. β_H we get, (with slight abuse of notation)

$$\xi_H \equiv -\frac{dR^e(\alpha)}{d\beta_H}\frac{\beta_H}{R^e(\alpha)} = \frac{1-\rho}{\rho}\frac{\beta_H}{1-\beta_H}$$

$$\Rightarrow \quad \beta_H^e = \rho$$

Similarly, for any fixed $(1-a_{H-2})$, we can write $\alpha_{H+1} = \frac{(1-\beta_H)}{N-H}(1-\beta_{H-1})(1-a_{H-2})$. Agent $H-1$ chooses β_{H-1} by taking into account optimal choice by H, i.e., using the fact that $\beta_H = \rho$. Hence, $\alpha_{H+1} = \frac{1-\rho}{N-H}(1-\beta_{H-1})(1-a_{H-2})$. Differentiating equation (20) w.r.t. β_{H-1} we get,

$$\xi_{H-1} = \frac{(1-\rho)}{\rho}\frac{\beta_{H-1}}{1-\beta_{H-1}}$$

$$\Rightarrow \quad \beta_{H-1}^e = \rho$$

Suppose that the statement is true for $h = H, H-1, \ldots, H-k+1$. I now show that the statement is true for $h = H-k$. For any given $(1-a_{H-k-1})$, we can write,

$$\alpha_{H+1} = \frac{1}{N-H}\prod_{h=0}^{k}(1-\beta_{H-h})(1-a_{H-k-1}) = \frac{(1-\rho)^k}{N-H}(1-\beta_{H-k})(1-a_{H-k-1})$$

Differentiating equation (20) w.r.t. β_{H-k} gives us

$$\xi_{H-k} = \frac{(1-\rho)}{\rho}\frac{\beta_{H-k}}{1-\beta_{H-k}}$$

$$\Rightarrow \quad \beta_{H-k}^e = \rho$$

A.III Proof of Proposition 5

Proving the statement is equivalent to proving that $\beta_h^e = \min\{\rho, \frac{1}{N-h+1}\}$. I first prove the following lemma:

Lemma 9 *In equilibrium, $H+1 \in \operatorname{argmin}_h\{\pi(\alpha_h^e R^e(\alpha^e))\}$.*

Proof: Suppose not. Then $\operatorname{argmin}_h\{\pi(\alpha_h^e R^e(\alpha^e))\}$ must be single valued. Suppose not. Then, there exists $l < m < H+1$ such that $l, m \in \operatorname{argmin}_h\{\pi(\alpha_h^e R^e(\alpha^e))\}$, i.e., $\alpha_l^e = \alpha_m^e$. Optimality of $R^e(\alpha)$ implies

$$\alpha_l^e\pi^{'}(\alpha_l^e R^e(\alpha^e)) = \frac{c}{N}$$

Since m chooses after l, the equation above implies that $\frac{dR^e(\alpha^e)}{d\alpha_m} = 0$ for $\alpha_m \geq \alpha_m^e$. Hence, $\alpha_m = \alpha_m^e$ can not be optimal.

Let $k = \text{argmin}_h\{\pi(\alpha_h^e R^e(\alpha^e))\}$ for some $k < H+1$. Then,

$$\alpha_k^e \pi'(\alpha_k^e R^e(\alpha^e)) = \frac{c}{N}$$

Differentiating the equation w.r.t. α_k at $\alpha_k = \alpha_k^e$ we get,

$$\rho\left[1 + \frac{dR^e(\alpha^e)}{d\alpha_k}\frac{\alpha_k^e}{R^e(\alpha^e)}\right] = 1$$

$$\Rightarrow \quad \frac{dR^e(\alpha^e)}{d\alpha_k} = \frac{1-\rho}{\rho}\frac{R^e(\alpha^e)}{\alpha_k^e} > 0$$

Hence, $\alpha_k = \alpha_k^e$ is not optimal choice for k. □

Optimal choice of H: Lemma 9 implies

$$\alpha_{H+1}^e \pi'(\alpha_{H+1}^e R^e(\alpha^e)) = \frac{c}{N} \tag{21}$$

Fix $(1 - a_{H-1}^e)$. Then, by Lemma 8, $\beta_H^e = \rho$, if possible. Additionally, we must have, $\beta_H^e \geq \frac{1}{N-H+1}$, since otherwise $\alpha_H^e < \alpha_{H+1}^e$. Hence, $\beta_H^e = \min\{\rho, \frac{1}{N-H+1}\}$.

Optimal choice of H − 1: Given $(1 - a_{H-2}^e)$ and the fact that β_H^e is constant, equation (21) implies $\beta_{H-1}^e = \rho$ if $\alpha_{H-1}^e > \alpha_{H+1}^e$, i.e., if

$$\beta_{H-1}^e(1 - a_{H-2}^e) > \frac{1}{N-H}(1-\beta_H^e)(1-\beta_{H-1}^e)(1-a_{H-2}^e)$$

For $\rho > \frac{1}{N-H+1}$, the inequality above holds. For $\rho \leq \frac{1}{N-H+1}$, the inequality above holds if $\rho > \frac{1}{N-H+2}$. Hence, $\beta_{H-1}^e = \min\{\rho, \frac{1}{N-H+2}\}$.

Iterating this logic for $H-2$, $H-3$ and so on proves the result.