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Strategy-proof interval-social choice correspondences over single-peaked domains *

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Abstract

We consider a social choice model where voters have single-peaked preferences over the alternatives that are aggregated to produce 'intervals' of fixed cardinality, L. This is applicable in situations where the alternatives can be arranged in a line (e.g. plots of land) and a contiguous set of these are required (e.g. a hospital or a school). We define interval-social choice correspondences (I-SCCs) on profiles of single-peaked preferences which select intervals. We extend single-peaked preferences to *intervals* using *responsiveness*. We show that *generalized median-interval rules* are the only *strategy-proof, anonymous* and *interval efficient* I-SCCs. An I-SCC is *interval efficient* if no other interval can make every voter strictly better-off. We show that replacing *interval efficiency* with a stronger notion, *Pareto efficiency*, characterizes a sub-class of these rules called the *t-th interval rules*.

JEL classification: D71

Keywords: social choice correspondence, single-peaked preferences, responsive, strategy-proofness, generalized median voter.

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1 Introduction

There are many voting situations over a one-dimensional policy space where a continguous set of alternatives needs to be chosen. Consider the following examples:

- Choosing plots of land: A facility like a hospital or school needs to be constructed which requires a set of 'connected' or contiguous *plots*. Here the individual plots can be seen as 'alternatives'.
- Choosing a committee: A committee needs to be selected from the set of 'citizen-candidates' on the basis of location where the candidates need to be neighbours (to minimize costs, for example).

In many cases, voters may have an incentive to lie about their preferences if they can obtain a better outcome. Therefore, it is imperative to design an aggregation rule which is immune to such manipulation. In this paper, we study *strategy-proof social choice correspondences* (*SCC*) which pick 'contiguous' subsets (or intervals) of fixed cardinality (which we call interval-SCCs or I-SCCs) in the domain of 'extended' single-peaked preferences.

The classic works on *strategy-proof social choice functions*, Gibbard (1973) and Satterthwaite (1975), show that the only rules which are *strategy-proof* on the unrestricted domain with more than three alternatives are dictatorial rules.[1] When preferences are single-peaked, Moulin (1980) showed that *generalized median voter rules* are the only *strategy-proof*, *anonymous* and *Pareto optimal* social choice functions.

In our model, voters have single-peaked preferences over the set of alternatives (as defined in Black (1948), Arrow (2012) and Moulin (1980)) but the final outcome may be a contiguous subset of those alternatives.[2] This framework is applicable to settings where the voter preferences over alternatives are required to make decisions on public goods over intervals whose cardinality may not be known *a priori*. It is natural to assume that preferences are single-peaked when the policy space is ordered or one-dimensional as shown in Hotelling (1929) and Downs (1957).[3]

[1]Similar results have been shown for SCCs by Pattanaik (1973), Gärdenfors (1976), Barberà et al. (1977), Kelly (1977), Feldman (1979a), Feldman (1979b), Feldman (1980), Sato (2008), and more generally in Özyurt and Sanver (2009). Barberà et al. (2001) and Ching and Zhou (2002) provide similar results for *strategy-proof* mechanisms in a cardinal setting, Schummer and Vohra (2002) for trees, and Border and Jordan (1983) for n-dimensional Euclidean space.

[2]Ballester and Haeringer (2011) provides a characterization of the single-peaked domain.

[3]See Thomson (1997) and Amorós (2002) for applications of single-peaked preferences to public goods model.

To define *strategy-proofness* for an I-SCC requires extending preferences over alternatives to preferences over L-intervals. We extend these preferences to all intervals of cardinality L by assuming that they are *responsive*. This requires that if an alternative a has been removed from an L-interval A and another alternative b has been added to create a new L-interval B, then interval A is preferred to interval B if and only if alternative a is preferred to alternative b. Various preference extensions used in the literature on social choice theory (Sato (2008)) and matching theory (Konishi and Ünver (2006)) satisfy this property (see Barberà et al. (2004) for a survey on various preference extensions).[4]

We characterize *generalized median interval rules* which assign $n-1$ fixed intervals of cardinality L (where n is the number of voters) and outputs an L-interval. The top-L intervals of n voters and the $n-1$ fixed intervals of are listed from left to right with respect to their lower-end points. These rules then pick the median interval which may not be the top-L interval of the median voter. These rules are the interval-versions of rules characterized in Moulin (1980) and coincide when $L = 1$.

We show that *generalized median interval rules* are the only *strategy-proof*, *anonymous* and *interval efficient* I-SCCs. The first two axioms are standard in the literature, however, the version of *strategy-proofness* we use is not a direct extension of the condition to intervals. This axiom is defined on I-SCCs which choose intervals but the manipulations made by voters are over alternatives. Therefore, the voters have more deviations than they would if they could only manipulate 'intervals'. Due to this, the proof of the main theorem does not follow directly from the result in Moulin (1980). Additional properties of the domain need to be proved in order to rule out these deviations.

The last axiom is a weaker, interval-variant of *Pareto efficiency* and can be stated as follows. *Interval efficiency* of the I-SCC requires that there should not exist any L-interval that makes all the voters strictly better-off compared to the outcome of the I-SCC. When we require outcomes to be Pareto efficient in addition to being interval efficient, we characterize *t-interval rules* which pick the t-th position interval from the set of top-L intervals when they are arranged from left to right.

Our rules can also be seen as interval-based SCC versions of the rules characterised in Moulin (1980) for social choice functions. The proof of the main theorem proceeds in two steps. We first show that if voters have single-peaked preferences which are *responsive* over intervals then these preferences are *single-peaked over intervals* as well. In the second step, we show that a *strategy-proof* and *interval efficient* I-SCC must be *top-L only*, i.e., it is invariant to changes in the preference

[4]All the extensions mentioned in Sato (2008) are *responsive on intervals*. We show this in the appendix.

profile made outside the top-L intervals of voters. Finally, we use Moulin (1980)'s result to characterize *generalized median-interval rules*.

The paper is organized as follows. Section 2 will describe the model and definitions. Section 3 and 4 presents the set of axioms and results respectively. We conclude in Section 5. Some additional results and proofs are provided in the Appendix.

2 The Model

The set of voters is $N = \{1, 2, \ldots, n\}$, and the set of alternatives is $X = \{a_1, a_2, \ldots, a_m\}$. The alternatives are arranged according to an ordering $<$ on X such that $a_1 < a_2 < \cdots < a_m$. We will denote by a_j and a_{j+1} as two consecutive alternatives according to $<$.

Voter preferences over alternatives: Each voter i's preference, P_i, is a *linear order* which is *single-peaked* on X, i.e., there exists a 'peak', $\tau(P_i)$, such that for any $x, y \in X$,

$$\left[y < x \leq \tau(P_i) \text{ or } \tau(P_i) \leq x < y\right] \Rightarrow xP_iy,$$

where the peak, $\tau(P_i)$, is the top-ranked alternative in X for any voter $i \in N$.[5] Let $\mathcal{S}(X)$ be the set of all single-peaked preferences over X according to $<$ and let $P = (P_1, \ldots, P_n)$ denote a profile of preferences. Let $\mathcal{S}^n(X)$ be the set of all single-peaked profiles on X according to $<$.

Interval of cardinality L: For any $L \in \{1, 2, 3, \ldots, m\}$, $l \in \{1, \ldots, m - L + 1\}$ and $a_l \in X$, define an *interval of cardinality L* or *L-interval* as $[a_l] = \{a \in X \mid a_l \leq a < a_{l+L}\}$. Therefore $[a_l]$ includes all l alternatives from a_l to a_{l+L-1} according to the order $<$.

Ordering, $<_L$, over L-intervals: For any two intervals $[a_l]$ and $[a_r]$ of cardinality L, define an ordering $<_L$ over L-intervals as follows: $[a_l] <_L [a_r]$ if and only if $a_l < a_r$.

We say that two intervals $[a_l]$ and $[a_r]$ are *adjacent* if (i) $[a_l] \neq [a_r]$ and (ii) there is no alternative x such that $\min\{a_l, a_r\} < x < \max\{a_l, a_r\}$.

[5]A binary relation P defined on X is a *linear order* if it is (i) complete: either xPy or yPx $\forall x, y \in X$, (ii) transitive: $[xPy \text{ and } yPz] \Rightarrow [xPz]$ $\forall x, y, z \in X$ and (iii) antisymmetric: $[xPy \& yPx] \Rightarrow [x = y]$, $\forall x, y \in X$.

Example 1 *Consider the following example for $L = 3$. Three intervals are shown from left to right: $[a_1], [a_2]$ and so on till $[a_{m-2}]$. Interval $[a_1]$ is adjacent to $[a_2]$, $[a_2]$ is adjacent to $[a_3]$ and so on.*

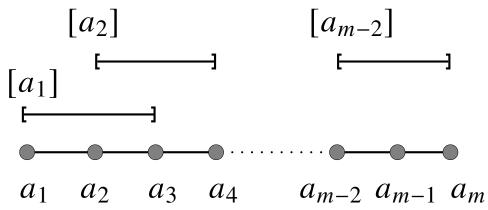

Figure 1: Alternatives, and intervals of cardinality 3

Claim 1 (*Top-L interval*:) *Take any $L \in \{1, ..., m\}$. The set of top-L ranked alternatives of any single-peaked preference P_i (denoted by P_i^L) for any $i \in N$ is an interval of cardinality L (henceforth, top-L interval).*

We prove Claim 1 by contradiction. It is trivially satisfied for $L \in \{1, m\}$. Suppose the set of top-L ranked alternatives P_i^L is not an interval, for some $L \in \{2, ..., m-1\}$. There exists distinct alternatives $x, y \in X \setminus \{\tau(P_i)\}$ such that (i) $x \in P_i^L$ and (ii) $y \notin P_i^L$, and either: (iii) $x < y < \tau(P_i)$ or (iv) $x > y > \tau(P_i)$. Since $y \notin P_i^L$ and $x \in P_i^L$, by definition of the top-L ranked set, xP_iy. This along with the fact that either (iii) or (iv) holds, is a contradiction to single-peakedness of P_i. Henceforth, we will use the term top-L interval to denote the set of top-L ranked alternatives in a preference.

We will denote the set of all non-empty subsets of cardinality L as $\mathcal{X}_L$ and the set of intervals of cardinality L as $\mathcal{I}_L$ for any $L \in \{1, \ldots, m\}$. Therefore, $\mathcal{I}_L \subset \mathcal{X}_L$. We fix the cardinality of intervals to be L throughout the length of this paper.

Interval-based Social Choice Correspondence (I-SCC)

In this paper, we only consider contiguous sets of cardinality L, i.e. L-intervals. An *interval-social choice correspondence (I-SCC)*, $f_I : \mathcal{S}^n(X) \to \mathcal{I}_L$ produces an interval $f_I(P) \in \mathcal{I}_L$ of cardinality L for every profile $P \in \mathcal{S}^n(X)$.

In order to compare the outcomes of I-SCCs with other outcomes, which are sets of cardinality L but not necessarily intervals, we need to extend voters' preferences over alternatives to subsets of fixed cardinality L. We define such extensions below.

Extension of preferences to $\mathcal{X}_L$

Extension of P_i: A weak order $\succsim_i$ for $i \in N$ defined over $\mathcal{X}_L$ is an *extension* of any $P_i \in \mathcal{S}(X)$. We refer to the top-ranked L-cardinality subset according to $\succsim_i$ as $\tau(\succsim_i)$. We impose a property *responsiveness* on intervals which is only applicable to the set $\mathcal{I}_L \subset \mathcal{X}_L$.

Responsiveness on intervals: Consider any two *adjacent* intervals $[a_l], [a_r] \in \mathcal{I}_L$ such that $a \in [a_l] \setminus [a_r]$ and $b \in [a_r] \setminus [a_l]$. Any extension $\succsim_i$ of P_i is *responsive on intervals* if,

$$\text{(i) } aP_ib \Leftrightarrow [a_l] \succ_i [a_r] \text{ and (ii) } bP_ia \Leftrightarrow [a_r] \succ_i [a_l]$$

where $\succ_i$ is the asymmetric part of $\succsim_i$. *Responsiveness over intervals* can also be interpreted as follows: if an alternative a is removed from an interval of cardinality L and another alternative b is replaced with it to create a new interval (thus making the two intervals adjacent) then the new interval is preferred over the old one if and only if b is strictly preferred over a and vice versa if a is strictly preferred over b. This version of responsiveness is similar to the one used in Bossert (1995) and is also used widely in the matching literature (Konishi and Ünver (2006)).

Imposing *responsiveness* over intervals is a weaker requirement than imposing it over all sets of cardinality L. However, we only require the weaker version since I-SCCs only produce intervals. We show that single-peaked preferences over X according to $<$ are *responsive over intervals* if and only if they are single-peaked over $\mathcal{I}_L$ according to $<_L$. We provide a formal proof of this below.

Single-peakedness over $\mathcal{I}_L$: A single-peaked preference $\succsim_i$ for voter $i \in N$ over $\mathcal{I}_L$ can be defined by replacing X with $\mathcal{I}_L$, $\tau(P_i)$ with $\tau(\succsim_i)$, and $<$ with $<_L$.

Proposition 1 *An extension, $\succsim_i$ over $\mathcal{X}_L$ of $P_i \in \mathcal{S}(X)$ is single-peaked over $\mathcal{I}_L$ (i.e. $\succsim_i \in \mathcal{S}(\mathcal{I}_L)$) with order $<_L$ and $\tau(\succsim_i)$ as the 'peak' if and only if it is responsive on intervals.*

Proof: We first show that *responsiveness* of $\succsim_i$ implies that it is single-peaked over $\mathcal{I}_L$. Consider an linear extension $\succsim_i$ of P_i that is *responsive on intervals*.

Denote the top-L interval of P_i by $[a_t]$. We will first show that for all $[a_{l-1}] <_L [a_l] \leq_L [a_t]$, we have $[a_l] \succ_i [a_{l-1}]$. Transitivity of $\succ_i$ will imply that $[a_t]$ is preferred to all the intervals on the 'left' according to $<_L$ and similar arguments for the intervals on the 'right' will then imply that $[a_t] = \tau(\succsim_i)$.

Case 1: Suppose $[a_l]$ and $[a_{l-1}]$ are two intervals such that $\tau(P_i) \in [a_l]$. Let $[a_t] = \{a_t, \dots, \tau(P_i), a_{r1}, a_{r2}, \dots, a_{rk}\}$ where $a_{r1}, \dots, a_{rk}$ are listed in the decreasing order of preference to the right of $\tau(P_i)$. Note that $[a_{t-1}] = ([a_t] \setminus \{a_{rk}\}) \cup \{a_{t-1}\}$. By *responsiveness* over $[a_t]$ and $[a_{t-1}]$, and due to the fact that a_{rk} belongs to top-L interval of P_i we have $a_{rk}P_i a_{t-1} \Rightarrow [a_t] \succ_i [a_{t-1}]$. Similarly by responsiveness on intervals, $[a_{t-1}]$ and $[a_{t-2}]$, we have $a_{rk-1}P_i a_{t-2} \Rightarrow [a_{t-1}] \succ_i [a_{t-2}]$.

Case 2: Consider $[a_l]$, $[a_{l-1}]$ such that $\tau(P_i) \notin [a_l]$. By single-peakedness of P_i, $a_{l-1} < a_{l+L-1} < \tau(P_i)$ implies $a_{l+L-1}P_i a_{l-1}$. By responsiveness $[a_l] \succ_i [a_{l-1}]$ since $[a_{l-1}] = ([a_l] \setminus \{a_{l+L-1}\}) \cup \{a_{l-1}\}$. By transitivity of $\succ_i$, for all $[a_l] \leq_L [a_t]$, $[[a_l] \succ_i [a_{l-1}]$ and $[a_{l-1}] \succ_i [a_{l-2}]] \Rightarrow [[a_l] \succ_i [a_{l-2}]]$. Repeated application of transitivity implies that $[a_l] \succ_i [a_{l-k}]$ for all $[a_l] \leq_L [a_t]$ and for all $k \leq l-1$.

Similar arguments can be made for intervals to the 'right' of $[a_t]$ according to $<_L$. Therefore $\succ_i$ is single-peaked on $\mathcal{I}_L$ with respect to $<_L$ and $\tau(\succsim_i) = [a_t]$.

We now show the converse. Consider an *extension* of P_i, $\succsim_i$ that is a linear order and single-peaked on $\mathcal{I}_L$ with respect to $<_L$ and $\tau(\succsim_i) = [a_t]$ where $[a_t]$ is the top-L interval of P_i.

To show *responsiveness on intervals* we need to show that for any two adjacent intervals $[a_l]$, $[a_{l-1}]$, we have $[[a_l] \succ_i [a_{l-1}]] \Leftrightarrow [a_{l+L-1}P_i a_{l-1}]$ and $[[a_{l-1}] \succ_i [a_l]] \Leftrightarrow [a_{l-1}P_i a_{l-L+1}]$. Consider intervals $[a_{l-1}]$ and $[a_l]$ on the 'left' of $[a_t]$ according to $<_L$ i.e. $[a_{l-1}] <_L [a_l] \leq_L [a_t]$.

Case I: Suppose $\tau(P_i) \notin [a_l]$. By single-peakedness of $\succsim_i$, we have $[a_l] \succ_i [a_{l-1}]$. Since $[a_{l-1}] = ([a_l] \setminus \{a_{l+L-1}\}) \cup \{a_{l-1}\}]$, by single-peakedness of P_i, $a_{l-1} < a_{l+L-1} < \tau(P_i)$ implies $a_{l+L-1}P a_{l-1}$.

Case II: $\tau(P_i) \in [a_l]$. For $[a_l] = [a_t]$, single-peakedness of $\succsim_i$ implies $[a_t] \succ_i [a_{t-1}]$ since $[a_{t-1}] = ([a_t] \setminus \{a_{t+L-1}\}) \cup \{a_{t-1}\}$. Similar arguments can be made for intervals on the right of $[a_t]$.

To define *generalized median interval rules*, we introduce some definitions:

(i) *Median:* Consider any integer $p > 0$ and a sequence of alternatives $B = (x_1, x_2, \dots, x_{2p-1})$. Repetitions are allowed and alternatives are arranged according to $<$. An alternative $x \in B$ is the *median* of this sequence, denoted by $med(x_1, x_2, ..., x_{2p-1})$, if

$$|\{x' \in B : x' \leq x\}| \geq p \text{ and } |\{x' \in B : x \leq x'\}| \geq p.$$

Note that the median of a sequence with $2p-1$ alternatives is the p-th alternative.

Similarly we can define the *median of a sequence of intervals*, $([a_1], \ldots, [a_{2k-1}])$, as $med([a_1], ..., [a_{2k-1}]) = [a_k]$ for any integer $k > 0$.

(ii) *Fixed Intervals:* For the rule, a sequence of $n-1$ fixed intervals are added to every profile of voters, $P \in \mathcal{S}^n(X)$. We denote the set of fixed intervals by $(\tau(\hat{\succsim}_1), \ldots, \tau(\hat{\succsim}_{n-1}))$.

Generalized median interval (GMI) rules

An I-SCC, $f_I : \mathcal{S}^n(X) \longrightarrow \mathcal{I}_L$, is a *GMI* rule if there exist $n-1$ fixed intervals such that for any $P \in \mathcal{S}^n(X)$,

$$f_I(P) = med(\tau(\succsim_1), \ldots, \tau(\succsim_n), \tau(\hat{\succsim}_1), \ldots, \tau(\hat{\succsim}_{n-1}))$$

where $\tau(\succsim_i)$ is the top-L interval of voter $i \in N$ and $\tau(\hat{\succsim}_i)$ for all $i \in \{1, ..., n-1\}$ are fixed intervals. Note that for a given *generalized median interval rule* the fixed intervals are defined independently of the profiles and remain fixed for all $P \in \mathcal{S}^n(X)$. Therefore, different sets of fixed intervals define different *GMI rules*.

The t-th interval rule: If the fixed intervals are allowed to take only the two extreme locations, i.e., $\tau(\hat{\succsim}_i) \in \{[a_1], [a_{m+L-1}]\}$ for all $i \in \{1, \ldots, n-1\}$, then these *GMI rules* are called *t-th interval rule*. These rules select the t-th voter's top-L interval from the sequence $((\tau(\succsim_1), \ldots, \tau(\succsim_n)))$. For example, if half of the fixed intervals are assigned to $[a_1]$ and the other half to $[a_{m+L-1}]$ (when the total number of intervals is odd) then the corresponding rule picks the median interval, i.e., $med(\tau(\succsim_1), \tau(\succsim_2), ..., \tau(\succsim_n))$.

Note that the median interval is according to the left most alternative in top-L intervals and may be the top-L interval of the median voter to $\tau(P_i)$ (except when $L = 1$ as in Moulin (1980)). We provide an example to illustrate.

Example 1: Suppose the set of voters is $N = \{1, 2, 3\}$, there are five alternatives which are arranged as follows: $a_1 < \cdots < a_5$ and $L = 3$. Consider the following preferences:

Voter 1: $a_2 P_1 a_3 P_1 a_4 P_1 a_5 P_1 a_1$,
Voter 2: $a_3 P_2 a_4 P_2 a_5 P_2 a_2 P_2 a_1$,
Voter 3: $a_4 P_3 a_3 P_3 a_2 P_3 a_1 P_3 a_5$.

Let f_I be the *GMI rule* with fixed intervals $\tau(\hat{\succsim}_1) = [a_1]$ and $\tau(\hat{\succsim}_2) = [a_3]$. By definition of *GMI rule*, $f_I(P) = med([a_1], [a_2], [a_2], [a_3], [a_3]) = [a_2]$ which is not the top-L interval of the median voter, i.e., $[a_3]$.

3 Axioms

We impose the following axioms on I-SCCs.

Anonymity: An I-SCC, f_I, satisfies *anonymity* if for every preference profile $P \in \mathcal{S}^n(X)$, and for each permutation σ of N, $f_I(P) = f_I(P^\sigma(X))$, where $P_i^\sigma = P_{\sigma(i)}$ for each $i \in N$.

Anonymity implies that the outcome of an I-SCC is independent of the identities of voters.

Strategy-proofness: An I-SCC, f_I, is said to be *strategy-proof* if for every $(P_i, P_{-i}) \in \mathcal{S}^n(X)$,

$$f_I(P_i, P_{-i}) \succsim_i f_I(P'_i, P_{-i}) \; \forall P'_i \in \mathcal{S}(X).$$

In other words, *strategy-proofness* states that unilateral deviations do not make a voter strictly better-off.

Since the outcome of I-SCCs are intervals, a natural extension of efficiency would be to compare intervals in $\mathcal{I}_L$ which we define below.

Interval efficiency: An I-SCC, f_I, is said to be *interval efficient* if for any $P \in \mathcal{S}^n(X)$ and any $[a_l] \in \mathcal{I}_L$,

$$[\exists\, j \in N \text{ s.t. } [a_l] \succ_j f_I(P)] \Rightarrow [\exists\, k \in N \text{ s.t. } f_I(P) \succ_k [a_l]].$$

An I-SCC satisfies *interval efficiency* if for any voter who can be made strictly better-off by any interval other than $f_I(P)$ there will be another voter who is made strictly worse-off by that interval. *Interval efficiency* can be interpreted as the interval version of *Pareto efficiency*. We can show that an I-SCC which is *strategy-proof* and interval efficient assigns the same interval to two profiles which have the same top-L intervals for all voters. We call this property *top-L only* which we defined below.

Top-L only: An I-SCC, f_I, is said to be *top-L only* if for all $P, P' \in \mathcal{S}^n(X)$ such that $\tau(\succsim_i) = \tau(\succsim'_i)$ for all $i \in N$, then $f_I(P) = f_I(P)$.

Proposition 2 *Suppose $f_I : \mathcal{S}^n(X) \to \mathcal{I}_L$ is strategy proof and interval efficient.*

Then it is top-L only.

Proof: Suppose f_I is a *strategy-proof* and *interval efficient* I-SCC. Since the top-L interval of every voter is an interval, we can denote it as $\tau(\succsim_i) = [P_i^l]$ where P_i^l is the left-most alternative in the top-L interval of the top-L interval of voter i for any $i \in N$.

We define $[\underline{a}, \overline{a}]$ as the smallest interval such that $\bigcup_{i\in N}[P_i^l] \subseteq [\underline{a}, \overline{a}]$. We show that any interval $[a_l]$ is *interval efficient* if and only if $[a_l] \subseteq ([\underline{a}, \overline{a}] \cap \mathcal{I}_L)$.
($\Rightarrow$) All voters prefer $[\underline{a}, \underline{a} + L - 1]$ to any other interval $[a_l] <_L ([\underline{a}, \underline{a} + L - 1]$, since their top-$L$ alternatives are on the right of $[a_l]$. Similarly, all voters prefer $[\overline{a} - L + 1, \overline{a}]$ to any other interval $[\overline{a} - L + 1, \overline{a}] <_L [a_l]$.

($\Leftarrow$) Suppose an L-interval $[a_l] \subset [\underline{a}, \overline{a}]$ is the outcome. Any distinct interval on the right of $[a_l]$ makes all voters i such that $[P_i^l] \leq_L [a_l]$ worse-off and any distinct interval on the left makes all voters i such that $[a_l] \leq_L [P_i^l]$ worse-off. Therefore, $f_I(P) \in [\underline{a}, \overline{a}]$ for all $P \in \mathcal{S}^n(X)$.

Pick any two profiles $P, P' \in \mathcal{S}^n(X)$ such that $[P_i^l] = [P_i'^l]$ for all $i \in N$. Consider the following sequence of profiles,

$P^0 = (P_1, P_2, P_3, \ldots, P_n),$
$P^1 = (P'_1, P_2, P_3, \ldots, P_n),$
$P^2 = (P'_1, P'_2, P_3, \ldots, P_n),$
$P^3 = (P'_1, P'_2, P'_3, \ldots, P_n),$
$\vdots$
$P^n = (P'_1, P'_2, P'_3, \ldots, P'_n).$

We show that for each $q \in \{0, \ldots, n-1\}$, the outcomes of consecutive profiles coincide i.e. $f_I(P^q) = f_I(P^{q+1})$ for all $q \in \{0, \ldots, n-1\}$. We will show the proof for $P^0 \to P^1$. Similar arguments can be made for other values of q.

Assume for contradiction that $f_I(P) = f_I(P^0) = [a_l] \neq f_I(P'_1, P_{-1}) = f_I(P^1) = [a_r]$. Assume w.l.o.g that $[P_1^l] = [P_1'^l] \leq_L [a_l]$.

Case 1: $[P_1'^l] \leq_L [a_l] <_L [a_r]$. Voter 1 can deviate at profile P^1 from P'_1 to P_1 and be better-off at the profile P^0 by single-peakedness of $\succsim_i$.

Case 2: $[a_r] <_L [a_l]$. There are two sub-cases:
Case 2.1: $[P_1^l] <_L [a_r] <_L [a_l]$. Then voter 1 at preference P_1 can deviate to P'_1 and be better-off by single-peakedness of $\succsim_i$.
Case 2.2: $[a_r] <_L [P_1^l] = [P_1'^l] <_L [a_l]$. Suppose $P_1 \in \mathcal{S}(X)$ is such that all the alternatives to the left of P_1^l are preferred over alternatives to the right of $P_1^l + L - 1$. In this case, voter 1 will deviate to $[P_1'^l]$.

Since f_I is *strategy-proof*, all the above arguments lead to a contradiction. Therefore $[a_r] = [a_l]$ and $f_I(P^0) = f_I(P^1)$. Similar arguments can be made for the case where $[a_l] \leq_L [P_1^l] = [P_1^{\prime l}]$. Therefore, $f_I(P) = f_I(P^0) = f_I(P^n) = f_I(P')$.

■

Therefore, the outcome of an *interval efficient* and *strategy-proof* I-SCC, f_I, depends only on the top-L intervals irrespective of the ordering of alternatives which belong to it. In other words, the outcome is same for two profiles which have the same set of top-L alternatives.

4 Results

Theorem 1 *Suppose the extension $\succsim_i$ of preferences P_i for each voter $i \in N$ is responsive on intervals. An I-SCC, $f_I : \mathcal{S}^n(X) \rightarrow \mathcal{I}_L$, is anonymous, strategy-proof and interval efficient if and only if it is a generalized median interval rule.*

The proof of the theorem follows from Propositions 1, Proposition 2 and Moulin (1980). We provide a sketch of the proof.

We prove necessity first. *GMI rules* are *anonymous* since the rule is invariant to permutation of voters' preferences. *GMI rules* are *interval efficient* since they always pick an L-interval which lies between the left-most and right-most top-L interval of voters. This implies that any other interval which makes a voter strictly better-off will also make a voter strictly worse-off. We show that *GMI rules* are *strategy-proof*. Since *GMI rules* only take into account the top-L interval of voters, a voter i has to change her own top-L interval to change the outcome. Since the *GMI rule* picks the median of the top-L intervals and the fixed intervals, the only way to change the outcome is to 'report' the top-L interval on the other side of the interval outcome, $f_I(P)$ (the outcome under truthful reporting). As a result of the deviation, the outcome moves further away from the 'true' top-L interval of voter i. Since the extension $\succsim_i$ of P_i is single-peaked over intervals (Proposition 1) any such deviation will make voter i worse-off.

We now show sufficiency of the axioms. By Proposition 1, a preference $\succeq_i$ over L-*intervals* is single-peaked according to $<_L$. This implies that intervals can be arranged from left to right and can be seen as 'interval-alternatives' in the relevant interval-based single-peaked domain. We have shown that any I-SCC which is *strategy-proof* and *interval efficiency* must be top-L only (Proposition 2). This implies that only the top-L intervals determine the outcome of such I-SCCs.

The next part of proof follows from Moulin (1980). Step one requires identifying the 'fixed intervals'. The first fixed interval is obtained by considering a profile where $n-1$ voters' top-L interval is $[a_1]$ and one voter's is $[a_{m-L+1}]$. Subsequent fixed intervals can be obtained by considering profiles where $n-j$ voters' top-L interval is $[a_1]$ and other j voters' top-L interval is $[a_{m-L+1}]$. *Interval efficiency* ensures that the outcome of profiles where top-L interval of all voters is $[a_1]$ (or $[a_{m-L+1}]$) is $[a_1]$ (or $[a_{m-L+1}]$). Once the fixed intervals are identified, induction is used over the number of voters whose top-L interval is neither $[a_1]$ nor $[a_{m-L+1}]$. At each step of induction, a *strategy-proof* I-SCC has to choose the median interval of $(\tau(\succsim_1), \ldots, \tau(\succsim_n), \tau(\hat{\succsim}_1), \ldots, \tau(\hat{\succsim}_{n-1}))$. Therefore any rule that satisfies the three axioms must be a *GMI rule*.

A stronger notion of efficiency on intervals can be imposed which requires that the outcome cannot be improved upon by *any subset of* $\mathcal{X}_L$ which makes all the voters strictly better-off. To define this notion of efficiency which we refer to as *Pareto efficiency*, we use the following extension.

Lexicographic Max:. Consider two sets $A, B \in \mathcal{X}_L$. Suppose that the alternatives in A and B are $a_1 P a_2 P \cdots P a_{L-1} P a_L$ and $b_1 P b_2 P \cdots P b_{L-1} P b_L$ respectively. The extension $\succsim$ of P to $\mathcal{X}_L$ is *lexicographic max* if,

$$A \succ B \Leftrightarrow \exists l \text{ s.t. } \begin{bmatrix} a_l P b_l \\ \forall m < l, \quad a_m = b_m \end{bmatrix}.$$

For comparing two sets A and B, all the alternatives in each set are arranged from best to worst in preference. If the first best alternative in one set is strictly preferred to the first best alternative in other then that set is strictly preferred. If the first best alternatives in the two sets are the same then it compares second best alternatives and ranks them accordingly and so on.[6]

Pareto efficiency: An I-SCC, f_I, is said to be *Pareto efficient* if for any $P \in \mathcal{S}^n(X)$ and any $A \in \mathcal{X}_L$,

$$[\exists\, j \in N \text{ s.t. } A \succ_j f_I(P)] \Rightarrow [\exists\, k \in N \text{ s.t. } f_I(P) \succ_k A].$$

Pareto efficiency states that for any voter j who can be made strictly better-off by any set other than A there is another voter k who is made strictly worse-off by that set. Interval efficiency is weaker than Pareto efficiency if imposed on the set of intervals $\mathcal{I}_L$. However, this is not the case in general if imposed on $\mathcal{X}_L$.[7]

Theorem 2 *Suppose the extension $\succsim_i$ of preferences P_i for each voter $i \in N$ is lexicographic max on $\mathcal{X}_L$. An I-SCC, $f_I : \mathcal{S}^n(X) \to \mathcal{I}_L$, is anonymous, strategy-proof and Pareto efficient if and only if it is a t-th interval rule.*

[6]Note that the *lexicographic max* extension is responsive on intervals. The proof is provided in the appendix.

[7]We provide an example in the Appendix to illustrate this.

The necessity of *Pareto efficiency* is implied by the fact that *t-th interval rules* always pick top-L interval of a voter which is a *Pareto efficient* outcome.

For sufficiency, note that if an I-SCC satisfies *Pareto efficiency* and *strategy-proofness* then it is *top-L only*. This is due to Proposition 2 and the fact that *Pareto efficiency* is stronger that *interval efficiency* in $\mathcal{I}_L$. Therefore an I-SCC that satisfies anonymity, *Pareto efficiency* and *strategy-proofness* will be a sub-class of *GMI rules*. We use the following claim to show sufficiency.

Claim 2 *The outcome of a Pareto efficient I-SCC, f_I, must be a top-L interval of a voter.*

Proof. Suppose $f_I(P)$ is not a top-L interval of any voter. Then there exist at least one top-L interval of a voter on the right and one on the left of $f_I(P)$.[8] Suppose for contradiction that there in no top-L interval to the right (left) of $f_I(P)$, then the right-most (left-most) top-L interval of voters is a Pareto improvement.

Therefore, let I_l and I_r be the top-L intervals closest to $f_I(P)$ according to $<$ on the left and right respectively. We can show that there is a Pareto improving subset of cardinality L (need not be an interval) over $f_I(P)$ which will make all the voters strictly better-off.

Replace the left end-point alternative of $f_I(P)$ with the left-most alternative in $f_I(P) \setminus I_r$ and right end-point alternative of $f_I(P)$ with the right-most alternative in $f_I(P) \setminus I_l$. Since the extended preferences are *lexicographic max*, the voter with top-L interval I_r strictly prefers the new set over $f_I(P)$ and so do all the voters with top-L interval to her right. Similarly for the voter with top-L interval I_l and others on her left are also strictly better-off. ■

Therefore, any I-SCC, f_I, that is *anonymous*, *strategy-proof* and *Pareto efficient* is a *GMI rule* which can only pick a top-L interval of a voter. This implies that f_I is a GMI rule where $\{\tau(\hat{\succsim}_i))\}$ is equal to either a_1 or $[a_{m-L+1}]$. Hence, f_I must be a *t-th interval rule*.

5 Conclusion

We characterize *generalized median interval rules* on an extended single-peaked domain which satisfy responsiveness on intervals. It remains to be seen what the class of *strategy-proof* and *Pareto efficient non-interval* SCCs are. The answer to this question will depend on the nature of assumptions made on preference

[8]Note that we can consistently use the term 'left' and 'right' for L-intervals as before. The proof does not depend on the location of voters' peaks within the top-L intervals.

extensions to non-interval sets of cardinality L.

6 Appendix

We provide definitions of extensions in Sato (2008) when restricted to comparing L-cardinality subsets in $\mathcal{X}_L$[9]. Consider two sets A and B with alternatives $\{a_1, \ldots, a_L\}$ and $\{b_1, \ldots, b_L\}$ respectively.

(i) Lexicographic Max extension: Assume $a_1 P a_2 P \cdots P a_{L-1} P a_L$ and $b_1 P b_2 P \cdots P b_{L-1} P b_L$, then

$$A \succ^{top} B \Leftarrow \exists l \text{s.t.} \begin{bmatrix} a_l P b_l \\ \forall m < l, \quad a_m = b_m \end{bmatrix}$$

(ii) Lexicographic min extension: Assume $a_L P a_{L-1} P \cdots P a_2 P a_1$ and $b_L P b_{L-1} \cdots P b_2 P b_1$, then

$$A \succ^{bot} B \Leftrightarrow \text{ same as } \succ^{top}$$

(iii) Max extension: Assume $a_1 P a_2 P \cdots P a_{L-1} P a_L$ and $b_1 P b_2 P \cdots P b_{L-1} P b_L$, then

$$A \succ^{max} B \Leftrightarrow \exists l \text{s.t.} \begin{bmatrix} a_l P b_l \text{ or } \begin{bmatrix} a_l = b_l \\ a_{L-l+1} P b_{L-l+1} \end{bmatrix} \\ a_m = b_m \quad \forall m < l, \quad \forall m > L - l + 1 \end{bmatrix}$$

(iv) Min extension: assume $a_L P a_{L-1} P \cdots P a_2 P a_1$ and $b_L P b_{L-1} \cdots P b_2 P b_1$, then

$$A \succ^{min} B \Leftrightarrow \text{same as } \succ^{max}$$

Proposition 3 *All preference extensions defined above are responsive on intervals with same cardinality.*

Proof. Consider, any $[a_k] \in \mathcal{I}_L$. To show responsiveness on intervals we have to compare either right adjacent: $[a_k]$ to $[a_{k+1}]$ i.e. $[a_k] \setminus [a_{k+1}] = a_k$ and $[a_{k+1}] \setminus [a_k] = a_{k+L}$ or left adjacent: $[a_k]$ to $[a_{k-1}]$ i.e. $[a_k] \setminus [a_{k-1}] = a_{k+L-1}$ and $[a_{k-1}] \setminus [a_k] = a_{k-1}$.

[9]These definitions are defined in Sato (2008) for $\mathcal{X}$. Here we restrict them to $\mathcal{X}_L$.

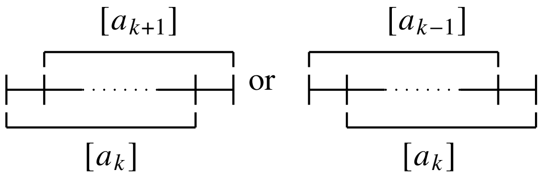

Figure 2: Adjacent intervals to $[a_k]$

Let's do for $\succsim^{top}$ and $\succsim^{max}$, similar arguments work for $\succsim^{bot}$ and $\succsim^{min}$. Let $[a_{k'}] \in \{[a_{k-1}], [a_{k+1}]\}$, arrange the elements according to P. Call the deleted alternative as a_s and added alternative as $a_{s'}$, where $s \in \{1, \dots, L\}$
$[a_k] = a_1 P a_2 P \dots a_s \dots a_L$
$[a_{k'}] = a_1 P a_2 \dots a_{s'} \dots a_L$ Note that $\succ^{top}$ will start comparing from a_1 using P, if the $a_s P a_{s'}$, we will have $[a_k] \succ^{top} [a_{k'}]$ and if $a_{s'} P a_s$, we will have $[a_{k'}] \succ^{top} [a_k]$. Similarly for $\succ^{max}$, if $a_s P a_{s'}$ either it occurs in max comparison or min comparison $[a_k] \succ^{top} [a_{k'}]$ and vice versa. ■

Example 2: Consider $N = \{1, 2\}$, $X = \{a_1, a_2, a_3, a_4\}$ and $L = 2$. The preferences of voters are $a_1 P_1 a_2 P_1 a_3 P_1 a_4$ and $a_4 P_2 a_3 P_2 a_2 P_2 a_1$ which are single-peaked according to $a_1 < a_2 < a_3 < a_4$. We consider the following *lexicographic max* extension $\succsim_i^{top}$ as defined above in Proposition 3.

$$[A \succ_i^{top} B] \Leftrightarrow [\exists\ s \text{ such that } x_s P_i y_s \text{ and } x_t = y_t \text{ for all } t < s]\,.$$

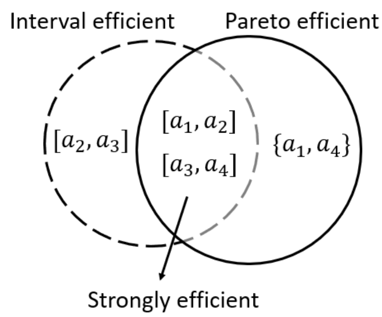

Figure 3: Efficient outcomes in $\mathcal{X}_2$ for Example 2

We show that $[a_1, a_2], [a_3, a_4] \in \mathcal{I}_2$ are *interval efficient* and *Pareto efficient*. As compared to $[a_1, a_2]$, the interval $[a_2, a_3]$ makes voter 2 strictly better-off but voter 1 strictly worse-off. Similar argument holds for $[a_3, a_4]$. Hence $[a_1, a_2]$ is *interval efficient*. It is easy to verify that $[a_3, a_4]$ is also *interval efficient*. There is no Pareto improvement over $[a_1, a_2]$ ($[a_3, a_4]$) as it the top-2 interval for voter 1 (voter 2). This is due to the fact that any other set in $\mathcal{X}_2$ will make voter 1 (voter 2) worse-off.

It is easy to verify that $[a_2, a_3]$ is *interval efficient* but not *Pareto efficient* as $\{a_1, a_4\} \in \mathcal{X}_2$ is strictly better for both the voters.

Therefore, there are *Pareto efficient* sets which are non-intervals but can Pareto dominate *interval efficient* intervals. Also notice that *if an interval is Pareto efficient then it is interval efficient* because the set of possible interval improvements is a subset of possible Pareto improvements.

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