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Strategy-proof interval-social choice correspondences over extended single-peaked domains *

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Abstract

We consider a social choice model where voters have single-peaked preferences over the alternatives that are aggregated to produce contiguous sets or *intervals* of fixed cardinality, L. This is applicable in situations where the alternatives can be arranged in a line (e.g. plots of land) and a contiguous subset of these is required (e.g. a hospital or a school). We define interval-social choice correspondences (I-SCCs) on profiles of single-peaked preferences which select intervals. We extend single-peaked preferences to *intervals* using *responsiveness*. We show that *generalized median-interval (GMI)* rules are the only *strategy-proof, anonymous* and *interval efficient* I-SCCs. These rules are interval versions of generalized median voter rules which consist of the median, min and max rules. We show that responsiveness over intervals is necessary for the strategy-proofness of the GMI rule if preferences over alternatives are single-peaked.

JEL classification: D71

Keywords: social choice correspondence, interval, single-peaked preferences, responsive, strategy-proofness, median voter.

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1 Introduction

There are many voting situations over a one-dimensional policy space where a contiguous set of alternatives of a fixed cardinality (say, L) needs to be chosen. We will call these contiguous sets of cardinality L, *intervals* or more specifically, *L-intervals*. Consider the following examples:

- Choosing L plots of land: Individuals have preferences over single plots of land or 'alternatives' which are ordered on a line. A public good like a hospital or school needs to be constructed which requires L number of contiguous *plots* or an L-interval.

- Choosing a committee with L members: A committee needs to be selected from the set of 'candidates' which are located on a line, and a connected set of L candidates need to be chosen (to reduce costs, for example).

We consider aggregators or social choice correspondences which only pick intervals of cardinality L (*interval-social choice correspondences (I-SCCs)*). In many cases, voters may have an incentive to lie about their preferences if they can obtain a better outcome. Therefore, it is imperative to design an aggregation rule which is immune to such unilateral manipulations or is *strategy-proof*. In this paper, we study *strategy-proof* aggregation of 'extended' single-peaked preferences which pick L-intervals.

In our model, voters have single-peaked preferences over the set of alternatives (as defined in Black (1948), Arrow (2012)) which will be extended to L-intervals. It is natural to assume that preferences are single-peaked when the policy space is ordered or one-dimensional as shown in Hotelling (1929) and Downs (1957).[1] Since L-intervals are chosen, we will extend preferences of voters over alternatives to obtain preferences over L-intervals using *responsiveness over intervals*. Responsiveness over intervals requires that if an alternative a has been removed from an L-interval A and another alternative b has been added to create a new L-interval B, then interval A is preferred to interval B if and only if alternative a is preferred to alternative b. We provide an example.

Example 1 *Suppose there are five alternatives $a_1 < a_2 < a_3 < a_4 < a_5$ and let $L = 3$. There are three intervals of cardinality 3 from left to right: $[a_1] = \{a_1, a_2, a_3\}$, $[a_2] = \{a_2, a_3, a_4\}$ and $[a_3] = \{a_3, a_4, a_5\}$ where we denote $[a_l]$ as the L-interval where a_l is the left-end point of the interval, for any $l \in \{1, 2, 3\}$.*[2]

[1]See Thomson (1997) and Amorós (2002) for applications of single-peaked preferences to public goods model.

[2]One could use the last element of the interval as well to denote intervals. This does not affect the analysis in any way.

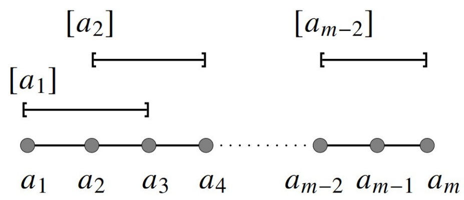

Figure 1: Intervals of cardinality 3

We now illustrate responsiveness of preferences. Suppose a voter i has the following single-peaked preference over alternatives, P_i: $a_2 P_i a_3 P_i a_1 P_i a_4 P_i a_5$. Therefore, peak of voter i is a_2, and preference is decreasing on either side of the peak. Responsiveness over intervals requires that interval $[a_1] = \{a_1, a_2, a_3\}$ is preferred to interval $[a_2] = \{a_2, a_3, a_4\}$ if and only if the only element in $[a_1]$ which is not in $[a_2]$, i.e. a_1, is better than the only element in $[a_2]$ which is not in $[a_1]$, i.e. a_4. Since $a_1 P_i a_4$, responsiveness implies that voter i must prefer $[a_1]$ over $[a_2]$. Note that responsiveness does not put any restrictions on intervals which differ by more than two alternatives, for example, $[a_1]$ and $[a_3]$. The lexicographic max preference extension over intervals works as follows: the alternatives in each interval are listed in descending order of preference: in $[a_1] : a_2 P_i a_3 P_i a_1$, and in $[a_3] : a_3 P_i a_4 P_i a_5$ and each corresponding ranked alternative is compared one by one till the lowest k-th ranked alternative in one sequence that is strictly better than its k-th ranked counterpart alternative in the other sequence (if all the alternatives ranked higher than the k-th ranked alternative in the first sequence are the same as the corresponding ranked alternative in the other sequence). In the above example, since the best alternative in $[a_1]$, i.e. a_2, is strictly better than the best ranked alternative in $[a_3]$, i.e. a_3, the max preference extension would rank $[a_1]$ higher than $[a_3]$. Therefore, responsiveness over intervals is weaker than assuming lexicographic max preference extension over intervals.[3] *We only use this property to compare adjacent intervals: two intervals are adjacent if their left end-points are adjacent (e.g. $[a_1]$ and $[a_2]$ above are adjacent). Various preference extensions used in the literature on social choice theory (Bossert et al. (2000), Pattanaik and Peleg (1984), Bossert (1995), Sato (2008)) and matching theory (Roth (1985) and Alcalde and Barberà (1994)) satisfy responsiveness.*[4]

This framework is applicable to settings where the voter preferences over alternatives can be used to make decisions on public goods over intervals whose cardinality is fixed throughout. Another advantage of this framework is that the cardinality of the interval may not be known *a priori*. Once the cardinality of the interval is known, the aggregation of preferences over intervals can be done using the preferences over alternatives. In this paper, we will keep the cardinality of the interval to be fixed at L. Therefore, voters only need to report their preferences over alternatives. An I-SCC is *strategy-proof* if no voter can benefit by misreporting her preference over the set of alternatives.

[3]One can check that all lexicographic max extensions are responsive over intervals but the converse is not true.

[4]See Barberà et al. (2004) for a survey on various preference extensions. All the extensions mentioned in Sato (2008) are *responsive* when restricted to intervals.

The classic works on *strategy-proof social choice functions*, Gibbard (1973) and Satterthwaite (1975), show that the only rules which are *strategy-proof* on the unrestricted domain with more than three alternatives are dictatorial rules. Similar results have been shown for SCCs in many works (see Gärdenfors (1976), Barberà et al. (1977), Kelly (1977), Sato (2008), and Özyurt and Sanver (2009).) Barberà et al. (2001) and Ching and Zhou (2002) provide similar results for *strategy-proof* mechanisms in a cardinal setting, Schummer and Vohra (2002) for trees, and Border and Jordan (1983) for the n-dimensional Euclidean space. When preferences are single-peaked, Moulin (1980) showed that *generalized median voter rules* are the only *strategy-proof*, *anonymous* and *Pareto optimal* social choice functions.[5] Our results generalize the results of the latter to the setting with L-intervals.

There are works which study similar social choice problems on a set of alternatives that is a subset of Euclidean space. Klaus and Protopapas (2020a) considers a model where the set of alternatives is $[0, 1]$. In their model, the preferences are single-peaked and based on absolute distance from peak. The extension to sets uses comparisons of the best and worst elements. A similar extension is used in Klaus and Protopapas (2020b) to characterize target-set correspondences. Klaus and Storcken (2002) studies a multidimensional model where the preferences over alternatives are single-peaked with a best point and *separable-quadratic* with respect to distance from the best point. However, all these papers study social choice over a connected subset of a Euclidean space. In our paper, we consider the set of discrete and finite set of alternatives.

Another paper which studies interval social choice in a discrete domain is Caramuta (2010). They consider two types of preferences: separable and additive. Their result is an extension of Barberà et al. (1991)'s, They obtain a negative result (dictatorial s.c.f.) with additivity, and a positive result with separability, where they characterize the interval variant of *voting by committees* rule. Our result adds to the literature on strategy-proof SCCs in a restricted domain- extended single-peaked domains.

Our first result (Proposition 1) states that when voters have single-peaked preferences over alternatives then their extended preferences to intervals will be *single-peaked over intervals* if and only if they are *responsive over intervals*. This implies that there is a peak interval and other intervals which are further away from this interval are strictly worse. This is an important insight in this domain which allows us to list the intervals from left to right. This result is proved using the fact that the 'peak- interval' or top-ranked interval is the set of top-L ranked alternatives in a voter's preference. Responsiveness then implies that interval adjacent to it on the left (or right) must be less preferred to the peak interval since the alternatives further away are strictly worse.

We characterize *generalized median interval (GMI)* rules which assign $n - 1$ fixed intervals of cardinality L (where n is the number of voters) and outputs an L-interval. The top-L intervals of n voters and the $n - 1$ fixed intervals are listed from left to right with respect to their lower-end points. These rules then pick the median interval which may not be the top-L interval of the median voter. These rules are the

[5] Barberà et al. (1993) provides a generalization of this result to the n-dimensional 'box' space.

interval-versions of rules characterized in Moulin (1980) and coincide with it when $L = 1$.

We show that GMI rules are the only *strategy-proof*, *anonymous* and *interval efficient* I-SCCs (Theorem 1. The first two axioms are standard in the literature, however, the version of *strategy-proofness* we use is not a direct extension of the condition to intervals. This axiom is defined on I-SCCs which choose intervals but the manipulations made by voters are over alternatives. Therefore, the voters have more deviations than they would if they could only manipulate 'intervals'. Due to this, the proof of the main theorem does not follow directly from the result in Moulin (1980). Additional properties of the domain need to be proved in order to rule out these deviations. In fact, these additional arguments arise due to the fact that voters report preferences over alternatives and not intervals. The last axiom is a weaker, interval-variant of Pareto efficiency and can be stated as follows. *Interval efficiency* of the I-SCC requires that there should not exist any L-interval that makes all the voters strictly better-off compared to the outcome of the I-SCC.

The proof of the first theorem proceeds in two steps. We first use Proposition 1 which provides preferences of voters over intervals. We then show that a *strategy-proof* and *interval efficient* I-SCC must be *top-L only* (Proposition 2). This implies that such I-SCCs are invariant to changes in the preference profile made outside the top-L intervals of voters. We first identify the fixed intervals used by the GMI rules using profiles where voters have preferences at the end points of the policy interval. We use induction on the number of voters who do not have such preferences to show that the rule must pick the median of the top-intervals and the fixed intervals across all profiles. The final arguments proceed by contradiction: if the I-SCC is not the GMI rule defined in the earlier steps, we construct preference profiles where a voter can deviate beneficially.

Finally, we show the necessity of responsiveness over intervals for the strategy-proofness of the GMI rule. This holds under the condition that voter preferences over alternatives is single-peaked. We show that if there exists any preference which is not consistent with single-peaked preferences over intervals, then we can find a GMI rule with specified fixed intervals and a profile over which it is not strategy-proof. This validates the argument that responsiveness of intervals is a fairly weak assumption over preference extensions over L-intervals. The paper is organized as follows. Section 2 will describe the model and definitions. Section 3 and 4 presents the set of axioms and results respectively. We conclude in Section 5.

2 The Model

The set of voters is $N = \{1, 2, \ldots, n\}$, and the set of alternatives is $X = \{a_1, a_2, \ldots, a_m\}$. The alternatives are arranged according to an ordering $<$ on X such that $a_1 < a_2 < \cdots < a_m$. We will denote by a_j and a_{j+1} as two consecutive alternatives according to $<$.

Voter preferences over alternatives: Each voter i's preference, P_i, is *single-peaked* on X, i.e., there exists a 'peak', $\tau(P_i)$, such that for any $x, y \in X$,

$$\left[y < x \leq \tau(P_i) \text{ or } \tau(P_i) \leq x < y\right] \Rightarrow xP_i y,$$

where the peak, $\tau(P_i)$, is the top-ranked alternative in X for any voter $i \in N$. Moreover, we require that P_i is a linear order on X.[6] Let $\mathcal{S}$ be the set of all single-peaked preferences over X according to $<$ and let $P = (P_1, \ldots, P_n)$ denote a profile of single-peaked preferences where each $P_i \in \mathcal{S}$. Let $\mathcal{S}^n$ be the set of all single-peaked profiles on X. We only consider aggregation rules which pick contiguous sets of cardinality L or L-intervals which we define below.

Interval of cardinality L: For any $L \in \{1, 2, 3, \ldots, m\}$ we define an *interval of cardinality* L or L*-interval* as $[a_l] = \{a \in X \mid a_l \leq a \leq a_{l+L-1}\}$ where $l \in \{1, \ldots, m-L+1\}$. For example, if $X = \{a_1, a_2, a_3, a_4, a_5\}$, then the interval $[a_2]$ for $L = 3$ and $L + l - 1 = 3 + 2 - 1 = 4$ is the set, $[a_2] = \{a_2, a_3, a_4\}$. Therefore $[a_l]$ includes all l alternatives from a_l to a_{l+L-1} according to the order $<$. We denote the set of all intervals of cardinality L over X as $\mathcal{I}_L$ for any $L \in \{1, \ldots, m\}$. We fix the cardinality of intervals to be L throughout the rest of this paper. For simplicity, we will refer to L-intervals as just intervals.[7]

Ordering, $<_L$, over L-intervals: For any two intervals $[a_l]$ and $[a_r]$, define an ordering $<_L$ as follows: $[a_l] <_L [a_r]$ if and only if $l < r$. Therefore, in the Example 1 with 5 alternatives and $L = 3$, we have $[a_1] < [a_2] < [a_3]$. Any two intervals $[a_l]$ and $[a_{l+1}]$ are *adjacent* for any $l \in \{1, 2, ..., m - L + 1\}$, i.e., the starting points of the two L-intervals are at the l-th and $(l + 1)$-th position respectively.

Three intervals are shown from left to right in Figure 1: $[a_1]$, $[a_2]$ and so on till $[a_{m-2}]$ for a given length of $L = 3$. Interval $[a_1]$ is adjacent to $[a_2]$, $[a_2]$ is adjacent to $[a_3]$ and so on. Given single-peaked preference P_i for any $i \in N$ we show that the top-L ranked alternatives in P_i are L-intervals. We prove this claim formally.

Claim 1 (*Top-L interval*:) *Take any $L \in \{1, ..., m\}$. The set of top-L ranked alternatives of any single-peaked preference P_i (denoted by P_i^L) for any $i \in N$ is an interval of cardinality L (henceforth, top-L interval).*

Proof. We prove Claim 1 by contradiction. It is trivially satisfied for $L \in \{1, m\}$. Suppose the set of top-L ranked alternatives $P_i^L = \{x | \#\{y : yR_i x\} \leq L\}$ is not an interval, for some $L \in \{2, ..., m - 1\}$. Then, there exist distinct alternatives $x, y \in X \setminus \{\tau(P_i)\}$ such that $x \in P_i^L$ and $y \notin P_i^L$, and either: (i) $x < y < \tau(P_i)$ or (ii) $x > y > \tau(P_i)$. Since $y \notin P_i^L$ and $x \in P_i^L$, by definition of the top-L ranked set, $xP_i y$. This along

[6] A binary relation P defined on X is a *linear order* if it is (i) complete: either xPy or yPx $\forall x \neq y$, (ii) transitive: $[xPy$ and $yPz] \Rightarrow [xPz]$ $\forall x, y, z \in X$ and (iii) asymmetric: $[xPy] \Rightarrow \neg[yP_i x]$, $\forall x, y \in X$.

[7] If intervals are represented using the right end points instead of left end points, our analysis would not change.

with the fact that either (i) or (ii) holds, implies that yP_ix. This is a contradiction since P_i is asymmetric. Therefore, P_i^L is an interval for all $i \in N$. Henceforth, we will use the term *top-L interval* to denote the set of top-L ranked alternatives, P_i^L for all $i \in N$. We are now ready to define aggregation rules.

An *interval-social choice correspondence (I-SCC)*, $f : \mathcal{S}^n \to \mathcal{I}_L$ produces an L-interval $f(P) \in \mathcal{I}_L$ for every profile $P \in \mathcal{S}^n$. In order to compare the outcomes of I-SCCs with other outcomes which are sets of cardinality L, we need to extend voters' preferences over alternatives to subsets of fixed cardinality L. We define preference extensions below.

Extension of preferences to $\mathcal{I}_L$

Extension of P_i: A weak order $\succsim_i$ for $i \in N$ defined over $\mathcal{I}_L$ is an *extension* of $P_i \in \mathcal{S}$.[8] We refer to the top-ranked L-cardinality subset according to $\succsim_i$ as P_i^L. Note that if $[a_l]$ and $[a_{l+1}]$ are two intervals then $[a_l] = ([a_{l+1}] \setminus \{a_{l+L}\}) \cup \{a_l\}$ and $[a_{l+1}] = ([a_l] \setminus \{a_l\}) \cup \{a_{l+L}\}$. In Example 1, $[a_1]$ and $[a_2]$ are adjacent, where $[a_1] = ([a_2] \setminus \{a_4\}) \cap \{a_1\}$ and $[a_2] = ([a_1] \setminus \{a_1\}) \cap \{a_4\}$. In other words, two intervals are adjacent if the left end point alternative (a_l) of an interval $[a_l]$ is removed from it, and another alternative right next to the right end point (a_{l+L}) is added to create a new interval $[a_{l+1}]$. We impose a property *responsiveness* on intervals which is only applicable to the adjacent intervals in $\mathcal{I}_L$.

Responsiveness on intervals: Consider any two *adjacent* intervals $[a_l], [a_{l+1}] \in \mathcal{I}_L$. Any extension $\succsim_i$ of P_i is *responsive on intervals* if,

$$\text{(i)} \quad a_l P_i a_{l+L} \iff [a_l] \succ_i [a_{l+1}], \textit{ and}$$
$$\text{(ii)} \quad a_{l+L} P_i a_l \iff [a_{l+1}] \succ_i [a_l].$$

Responsiveness over intervals can also be interpreted as follows: if an alternative a_l is removed from an interval $[a_l]$ and another alternative a_{l+L} is replaced with it to create a new interval $[a_{l+1}]$ (thus making the two intervals adjacent) then the new interval is preferred over the old one if and only if a_{l+L} is strictly preferred over a_l. This version of responsiveness is similar to the one used in Bossert (1995) and is also used widely in the matching theory literature (Roth (1985), Alcalde and Barberà (1994) etc.). Responsiveness is a fairly weak condition since it only imposes restrictions on adjacent intervals. For example, for a preference extension $\succsim_i$ of a single-peaked preference P_i with $L = 3$, we can have $[a_1] \succ_i [a_2]$ if and only if $a_1 P_i a_4$. Responsiveness does not impose anything on two intervals $[a_1]$ and $[a_3]$. However, our next proposition will show that preference extensions $\succsim_i$ are single-peaked over $\mathcal{I}_L$ according to $<_L$ if and if they are *responsive over intervals*. Since P_i is strict or asymmetric, by responsiveness, the induced extension $\succsim_i$ will be anti-symmetric.[9] We define single-peakedness over intervals first.

[8]A binary relation $\succsim$ defined on X is a *weak order* if it is (i) complete: either $x \succsim y$ or $y \succsim x$ $\forall x, y \in X$ and (ii) transitive: $[x \succsim y$ and $y \succsim z] \Rightarrow [x \succsim z]$ $\forall x, y, z \in X$.

[9]A weak relation $\succsim$ on X is anti-symmetric if $[x \succsim y$ and $y \succsim x] \Leftrightarrow [x = y]$, for all $x, y \in X$.

Single-peakedness over intervals in $\mathcal{I}_L$: A preference extension $\succsim_i$ of P_i for voter $i \in N$ is *single-peaked over $\mathcal{I}_L$ according to $<_L$* if there exists a 'peak interval' $\tau(\succsim_i)$ such that for any two intervals $[a_k]$ and $[a_{k'}]$,

$$\left[\tau(\succsim_i) \leq_L [a_k] <_L [a_{k'}] \text{ or } [a_{k'}] <_L [a_k] \leq_L \tau(\succsim_i)\right] \implies [a_k] \succ_i [a_{k'}].$$

Single-peakedness over intervals is similar to single-peakedness over alternatives where a peak interval is at the top of the preference and intervals further away according to $<_L$ are strictly worse. Moreover, two intervals on different sides of the peak interval can be ranked in either way. Let $\mathcal{S}(\mathcal{I}_L)$ denote the set of single-peaked preference (extensions), $\succsim_i$, over intervals for any $i \in N$. Our next Proposition establishes a connection between single-peakedness over alternatives and single-peakedness over intervals.

Proposition 1 *A linear extension, $\succsim_i$, of a single-peaked preference $P_i \in \mathcal{S}$ is single-peaked over $\mathcal{I}_L$ according to $<_L$ with $\tau(\succsim_i) = P_i^L$ as the top interval if and only if it is responsive on intervals.*

Proof: We first show that *responsiveness* of $\succsim_i$ implies that it is single-peaked over $\mathcal{I}_L$. Consider a linear extension $\succsim_i$ of P_i that is *responsive on intervals*. We first show that the top-L interval of P_i is $\tau(\succsim_i) = P_i^L$ i.e. the set of top L ranked alternatives in P_i is also the top-ranked interval in $\succsim_i$. Let $P_i^L = [a_t]$ for some $t \in \{1, 2, ..., m-L+1\}$. We will first show that (i) $[[a_{l-1}] <_L [a_l] \leq_L \tau(\succsim_i) = [a_t]] \implies [[a_l] \succ_i [a_{l-1}]]$ for all $l \in \{1, 2, ..., t\}$ and (ii) $[\tau(\succsim_i) = [a_t] \leq_L [a_l] <_L [a_{l-1}]] \implies [[a_l] \succ_i [a_{l-1}]]$ for all $l \in \{t, ..., m - L + 1\}$. Transitivity of $\succ_i$ will imply that $\tau(\succsim_i)$ is preferred to all the intervals on the 'left' according to $<_L$ and similar arguments for the intervals on the 'right' will then imply that $\tau(\succsim_i) = P_i^L = [a_t]$. We provide arguments for part (i) above. Part (ii) can be proved similarly.

Case 1: For any $l \in \{1, .., t\}$, let $[a_{l-1}]$ and $[a_l]$ be two intervals such that $\tau(P_i) \in [a_l]$. Let $P_i^L = \{a_t, a_{t+1} \ldots, \tau(P_i), a_{r1}, a_{r2}, \ldots, a_{rk}\}$ where $a_{r1}, \ldots, a_{rk}$ are elements to the right of $\tau(P_i)$ and are listed in order according to $<$. Note that we can write $[a_{t-1}] = ([a_t] \setminus \{a_{rk}\}) \cup \{a_{t-1}\}$. By responsiveness over the following intervals: $P_i^L = [a_t]$ and $[a_{t-1}]$, and due to the fact that a_{rk} belongs to the top-L interval of P_i and a_{t-1} does not, we have $a_{rk} P_i a_{t-1}$. Therefore, $P_i^L = [a_t] \succ_i [a_{t-1}]$. Similarly by responsiveness on intervals, $[a_{t-1}]$ and $[a_{t-2}]$, we have $a_{rk-1} P_i a_{t-2} \Rightarrow [a_{t-1}] \succ_i [a_{t-2}]$.

Case 2: For any $l \leq t$, let $[a_{l-1}]$ and $[a_l]$ such that $\tau(P_i) \notin [a_l]$. By single-peakedness of P_i, $a_{l-1} < a_{l+L-1} < \tau(P_i)$ implies that $a_{l+L-1} P_i a_{l-1}$. By responsiveness $[a_l] \succ_i [a_{l-1}]$ since $[a_{l-1}] = ([a_l] \setminus \{a_{l+L-1}\}) \cup \{a_{l-1}\}$. By transitivity of $\succ_i$, for all $[a_l] \leq_L P_i^L = [a_t]$, we have $[[a_l] \succ_i [a_{l-1}]$ and $[a_{l-1}] \succ_i [a_{l-2}]] \Rightarrow [[a_l] \succ_i [a_{l-2}]]$. Repeated application of transitivity of $\succsim_i$ implies that $[a_l] \succ_i [a_{l-k}]$ for all $[a_l] \leq_L P_i^L$ for all $k \in \{1, ..., l-1\}$. Similar arguments can be made for intervals to the 'right' of P_i^L according to $<_L$. Therefore $\succsim_i$ is single-peaked on $\mathcal{I}_L$ with respect to $<_L$ and its peak interval is given by $\tau(\succsim_i) = P_i^L$.

We now show the converse. Consider an extension of P_i, $\succsim_i$ that is single-peaked on $\mathcal{I}_L$ with respect to $<_L$ and $\tau(\succsim_i) = P_i^L = [a_t]$ for some $t \in \{1, 2, ..., m - L + 1\}$. To show responsiveness on intervals, we need to show that for any two adjacent intervals $[a_l]$, $[a_{l-1}]$, we have $[[a_l] \succ_i [a_{l-1}]] \Leftrightarrow [a_{l+L-1} P_i a_{l-1}]$ and $[[a_{l-1}] \succ_i [a_l]] \Leftrightarrow [a_{l-1} P_i a_{l+L-1}]$ for any $l \in \{1, ..., m - L + 1\}$. Consider intervals $[a_{l-1}]$ and $[a_l]$ on the 'left' of $\tau(\succsim_i)$ according to $<_L$ i.e. $[a_{l-1}] <_L [a_l] \leq_L \tau(\succsim_i) = [a_t]$. This is without loss of generality since $[a_l]$ and $[a_{l-1}]$ are adjacent, which implies that either (i) $a_{l-1} < a_l \leq \tau(\succsim_i)$ or (ii) $\tau(\succsim_i) \leq a_{l-1} < a_l$. We provide arguments for part (i) above. There are two sub-cases:

Case I: Suppose $\tau(P_i) \notin [a_l]$. By single-peakedness of $\succsim_i$, we have $[a_l] \succ_i [a_{l-1}]$. Since $[a_{l-1}] = ([a_l] \setminus \{a_{l+L-1}\}) \cup \{a_{l-1}\}]$, by single-peakedness of P_i, $a_{l-1} < a_{l+L-1} < \tau(P_i)$ implies $a_{l+L-1} P a_{l-1}$. Case II: Suppose $\tau(P_i) \in [a_l]$. Single-peakedness of $\succsim_i$ implies $\tau(\succsim_i) \succ_i [a_l] \succ [a_{l-1}]$. By similar arguments as in Case I, $[a_{l-1} < a_l \leq \tau(P_i)] \implies [a_l P_i a_{l-1}]$. Therefore, $\succsim_i$ is responsive. Similar arguments can be made for intervals on the right of $\tau(\succsim_i)$. Similar arguments can be made for part (ii) $\tau(\succsim_i) \leq a_{l-1} < a_l$.

■

Proposition 1 provides an important insight into the nature of preference extensions $\succsim_i$ of single-peaked preferences P_i for any $i \in N$. It states that if voters have single-peaked preferences over alternatives, then *any* preference extension of P_i over the domain of intervals $\mathcal{I}_L$ is *single-peaked over intervals* if and only if it *responsive over intervals*. In other words, we show in the proof that there exists a unique L-sized interval or 'peak interval' which is top-ranked in $\succsim_i$ and is also the set of top L ranked alternatives. Henceforth, we will denote the 'top' interval $\tau(\succsim_i) = P_i^L$ as the set of top L ranked alternatives. Other intervals which are further away from the peak-interval on the same side of the peak interval are strictly worse. We provide an example below.

Example 2 *Suppose the set of voters is $N = \{1, 2, 3\}$, there are five alternatives which are arranged as follows: $a_1 < \cdots < a_5$ and $L = 3$. Consider the following preferences:*

P_1	P_2	P_3
a_2	a_3	a_4
a_3	a_4	a_3
a_4	a_5	a_2
a_5	a_2	a_1
a_1	a_1	a_5

By Proposition 1, the responsive preference extensions $(\succsim_1, \succsim_2, \succsim_3)$ on $\mathcal{I}_L$ will be single-peaked preferences, *Voter 1: $\tau(\succsim_1) = [a_2]$ since $[a_2] = \{a_2, a_3, a_4\}$ is the set of top 3 ranked alternatives of voter 1. Due to Proposition 1, we know that preferences over $\mathcal{I}_L$ are single-peaked. Therefore, $[a_2] \succ_1 [a_3]$ and $[a_2] \succ_1 [a_1]$. However, we impose no restriction on how $[a_1]$ and $[a_3]$ are to be compared as long as $\succsim_1$ is complete over the pair of intervals. Similarly, voter 2's peak is $\tau(\succsim_2) = [a_3]$ and her preference extension is such that $[a_3] \succ_2 [a_2] \succ_2 [a_1]$. Preference extension, $\succsim_3$, of voter 3 is such that $\tau(\succsim_3) = [a_2]$, $[a_2] \succ_3 [a_3]$, $[a_2] \succ_3 [a_1]$.*

We introduce some definitions to define *generalized median interval rules*:

Median of a sequence of alternatives: Consider any integer $p > 0$ and a sequence of alternatives $B = (x_1, x_2, \ldots, x_{2p-1})$ where repetitions are allowed and alternatives are arranged according to $<$. An alternative $x \in B$ is the *median* of this sequence, denoted by $med(x_1, x_2, ..., x_{2p-1})$, if

$$|\{x' \in B : x' \leq x\}| \geq p \text{ and } |\{x' \in B : x \leq x'\}| \geq p.$$

Note that the median of a sequence with $2p-1$ alternatives is the p-th alternative, and the order of the sequence does not matter. For example, $med(a_1, a_5, a_2, a_3, a_2, a_4, a_5) = med(a_1, a_2, a_2, a_3, a_4, a_5, a_5) = a_3$ where $p = 4$, and $2p - 1 = 2(4) - 1 = 7$ since there are four alternatives (weakly) above and below a_3 (including itself) when the alternatives are arranged in ascending order (with repetitions). The median can be found by enumerating the alternatives from left to right and picking the p-th alternative out of a total of $2p - 1$ alternatives.

Median of a sequence of intervals: Similarly, we can define the *median of a sequence of intervals*, $([x_1], ..., \leq_L [x_{2k-1}])$, as $med([x_1], ..., [x_{2k-1}]) = [x_k]$ for any integer $k > 0$. For example, $med([a_1], [a_2], [a_2], [a_3], [a_4]) = [a_2]$ since there are three intervals on either side of $[a_2]$ (counting itself twice).

Fixed Intervals: A sequence of $n - 1$ fixed intervals will be denoted by $[\alpha_1], \ldots, [\alpha_{n-1}]$. These will be added to the vector of top-L intervals of the voters to compute the outcome of our main rule, which we define now.

Generalized median interval (GMI) rules: An I-SCC, $f^\alpha : \mathcal{S}^n \to \mathcal{I}_L$, is a *GMI* rule if there exist $n - 1$ fixed intervals $\alpha = ([\alpha_1], \ldots, [\alpha_{n-1}])$ such that for any $P \in \mathcal{S}^n$,

$$f^\alpha(P) = med(\tau(\succsim_1), \ldots, \tau(\succsim_n), [\alpha_1], \ldots, [\alpha_{n-1}]).$$

GMI rules pick the median interval from the sequence consisting of the top-L interval of voters $\{\tau(\succsim_i)\}_{i=1}^n$ and the given fixed intervals $\{[\alpha_i]\}_{i=1}^{n-1}$. Note that for a given GMI rule the fixed intervals are defined independently of the profiles and remain fixed for all $P \in \mathcal{S}^n$. Therefore, different sets of fixed intervals define different GMI rules.

Consider Example 2 again. Let f^{α} be the following GMI rule with two fixed intervals: $[\alpha_1] = [a_1]$ and $[\alpha_2] = [a_3]$. For the preference extensions derived in Example 2, we have $f^{\alpha}(P) = med(\tau(\succsim_1), \tau(\succsim_2), \tau(\succsim_3), [\alpha_1], [\alpha_2]) = med([a_2], [a_3], [a_2], [a_1], [a_3]) = [a_2]$ since it is the third interval while enumerating from left (or right). Note that GMI rules only take into account the top-L intervals of voters. Note that the median interval is according to the left most alternative in top-L intervals and may not be the top-L interval of the median voter (except when $L = 1$ as in Moulin (1980)). We provide an example to illustrate.

Example 3 *Suppose $N = \{1, 2, 3\}$ and $X = \{a_1, a_2, ..., a_5\}$ where $a_1 < \cdots < a_5$ and $L = 3$. For the following preferences over alternatives, we derive single-peaked preference extensions on $\mathcal{I}_L$:*

P_1	P_2	P_3
a_2	a_3	a_4
a_3	a_4	a_3
a_4	a_5	a_2
a_5	a_2	a_1
a_1	a_1	a_5

By Proposition 1, we get the following single-peaked preference extensions $\{\succsim_1, \succsim_2, \succsim_3\}$:

Voter 1: $\tau(\succsim_1) = [a_2]$ with $[a_2] \succ_1 [a_3]$, $[a_2] \succ_1 [a_1]$, and for completeness we can have either $[a_1] \succ_1 [a_3]$ or $[a_3] \succ_1 [a_1]$, voter 2: $\tau(\succsim_2) = [a_3]$ with $[a_3] \succ_2 [a_2] \succ_2 [a_1]$ and voter 3: $\tau(\succsim_3) = [a_2]$ with $[a_2] \succ_3 [a_3]$, $[a_2] \succ_3 [a_1]$, and for completeness we can have either $[a_1] \succ_3 [a_3]$ or $[a_3] \succ_3 [a_1]$.

Note that even though voters 1 and 3 have different preferences over alternatives, they may have similar preferences over intervals. Suppose f^{α} is the GMI rule with fixed intervals $[\alpha_1] = [a_1]$ and $[\alpha_2] = [a_3]$. By definition of GMI rule, $f^{\alpha}(P) = med(\succsim_1, \succsim_2, \succsim_3, [\alpha_1], [\alpha_2]) = med([a_2], [a_3], [a_2], [a_1], [a_3]) = [a_2]$. The median voter according to the peaks is voter 2, however the outcome of the GMI rule is not her peak interval $[a_3]$, rather the outcome is the peak interval of voters 1 and 3, i.e., $[a_2]$.

We show in Section 4 that *strategy-proofness* and *interval efficiency* imply the top-L only property which means that these I-SCCs only take as input the top-L intervals of voters. However, GMI rules are not the only I-SCCs which are top-L only. We define some top-L only I-SCCs over the single-peaked domain.

Dicatorial I-SCC: A GMI rule, f^i is dictatorial if for all $P \in \mathcal{S}^n$, $f^i(P) = \tau(\succsim_i)$. Dictatorial rules pick the dictator's (voter i's) peak interval for all profiles.

Min (max) I-SCCs: A GMI rule, f^{min} (f^{max}) is a min (max) rule if for all $P \in \mathcal{S}^n$, $f^{min}(P) = min\{\tau(\succsim_i)\}_{i\in N}$ ($f^{max}(P) = max\{\tau(\succsim_i)\}_{i\in N}$), where $min\{.\}_{i\in N}$ ($max\{.\}_{i\in N}$) picks the interval with the smallest (largest) interval according to $<_L$. Min and max I-SCCs are a sub-class of GMI rules if $\alpha = ([a_1], ..., [a_1])$ for min and $\alpha = ([a_{m-L+1}], ..., [a_{m-L+1}])$ for max. A median I-SCC can also be defined as a GMI rule with $\alpha = (\underbrace{[a_1], ..., [a_1]}_{\frac{n-1}{2}}, \underbrace{[a_m], ..., [a_m]}_{\frac{n-1}{2}})$ when n is odd.[10] We now present the axioms.

3 Axioms

In this section we list the axioms which will characterize GMI rules.

Anonymity: An I-SCC, f, satisfies *anonymity* if for every preference profile $P \in \mathcal{S}^n$, and for each permutation σ of N, $f(P) = f(P^\sigma)$, where $P^\sigma = (P_{\sigma(1)}, ..., P_{\sigma(n)})$.

Anonymity implies that the outcome of an I-SCC is independent of the identities of voters. All the rules mentioned above except the dictatorial rules are anonymous.

Strategy-proofness: An I-SCC, f, is said to be *strategy-proof* if for every profile $(P_i, P_{-i}) \in \mathcal{S}^n$,

$$f(P_i, P_{-i}) \succsim_i f(P_i', P_{-i}) \quad \forall P_i' \in \mathcal{S}.$$

In other words, *strategy-proofness* states that unilateral deviations do not make a voter strictly better-off. Note that the deviations of voters are in terms of preferences over alternatives, and the outcomes are intervals. Since the outcome of I-SCCs are intervals, a natural extension of efficiency would be to compare intervals in $\mathcal{I}_L$ which we define below. All the rules mentioned above strategy-proof. The proof for GMI rules (including min and max) rules is provided in the proof of our main result.

Interval efficiency: An I-SCC, f, is said to be *interval efficient* if for any $P \in \mathcal{S}^n$ and any $[a_l] \in \mathcal{I}_L$,

$$[\exists\ j \in N \text{ s.t. } [a_l] \succ_j f(P)] \Rightarrow [\exists\ k \in N \text{ s.t. } f(P) \succ_k [a_l]].$$

An I-SCC satisfies *interval efficiency* if for any voter who can be made strictly better-off by any interval $[a_l]$ there will be another voter who is made strictly worse-off by that interval. *Interval efficiency* can be interpreted as the interval version of *Pareto efficiency*. The dictatorial rule is interval efficient since it always picks the top-interval of the dictator. GMI rules are interval efficient as will be proved in the proof of Theorem 1.

[10] A similar rule can be defined for n even using the left or right median.

4 Results

We first show that all *strategy-proof* and *interval efficient* I-SCC must be *top-L only*. The latter property is the interval version of the *tops-only property* commonly used in the social choice literature. This implies that only changes in the top L intervals of voters can affect the outcome of a strategy-proof and interval efficient I-SCC. We define top-L only.

Top-L only: An I-SCC, f, is said to be *top-L only* if for all $P, P' \in \mathcal{S}^n$ such that $\tau(\succsim_i) = \tau(\succsim_i')$ for all $i \in N$, $f(P) = f(P')$.

It states that if voter i reports P with the same set of top L-ranked alternatives (which is always an interval, by Proposition 1) as in P', then the outcomes under P and P' are the same.

Proposition 2 *Suppose $f : \mathcal{S}^n \to \mathcal{I}_L$ is strategy-proof and interval efficient. Then it is top-L only i.e. the outcome of an I-SCC only depends on the top-L alternatives of the voters.*

Proof: Suppose f is a *strategy-proof* and *interval efficient* I-SCC. We will denote a closed interval (of any cardinality) as $[a_j, a_k] = \{x \in X | a_j \leq x \leq a_k\}$ and an open interval as $(a_j, a_k) = \{x \in X \mid a_j < x < a_k\}$. We define $[\underline{a}, \overline{a}]$ as the smallest interval which contains the top-L intervals of all voters, i.e. (i) $\bigcup_{i \in N}\{P_i^L\} \subseteq [\underline{a}, \overline{a}]$ for some $\underline{a}, \overline{a} \in X$ and (ii) $\nexists\, \underline{a}', \overline{a}' \in (\underline{a}, \overline{a})$ such that $\bigcup_{i \in N}\{P_i^L\} \subseteq [\underline{a}', \overline{a}']$.

Lemma 1 *An interval $[a_l]$ is interval efficient if and only if $[a_l] \subseteq [\underline{a}, \overline{a}]$.*

Proof We prove necessity first. All voters prefer $[\underline{a}, \underline{a} + L - 1]$ to any other interval $[a_l] <_L [\underline{a}, \underline{a} + L - 1]$, since their top-$L$ alternatives are on the right of $[a_l]$. Similarly, all voters prefer $[\overline{a} - L + 1, \overline{a}]$ to any other interval $[a_l] >_L [\overline{a} - L + 1, \overline{a}]$. We prove sufficiency. Suppose an L-interval $[a_l] \subseteq [\underline{a}, \overline{a}]$ is the outcome. Any distinct interval on the right of $[a_l]$ makes all voters i such that $[P_i^L] <_L [a_l]$ strictly worse-off and any distinct interval on the left makes all voters i such that $[a_l] <_L [P_i^L]$ strictly worse-off. Therefore, $f(P) \in [\underline{a}, \overline{a}]$ for all $P \in \mathcal{S}^n$.

■

We now argue that f must be top-L only. Take any two profiles P and P' in $\mathcal{S}^n$, with the same top-L intervals, i.e., $P_i^L = P_i'^L$ for all $i \in N$. We show that $f(P) = f(P')$. We show construct a sequence of profiles where each voter's preference changes to P_i' sequentially but the outcome of the rule does not change between any two consecutive profiles. Consider the following sequence of profiles:

$P^0 = P = (P_1, P_2, P_3, \ldots, P_n)$, $P^1 = (P_1', P_2, P_3, \ldots, P_n)$, $P^2 = (P_1', P_2', P_3, \ldots, P_n)$,..., $P^n = P' = (P_1', P_2', P_3', \ldots, P_n')$.

In the above sequence, the profile P is transformed one step at a time to the profile P'. We show that $f(P^q) = f(P^{q+1})$ for all $q \in \{0, ..., n-1\}$. We first provide the argument for $q = 0$. Similar arguments can be made for other values of q. Assume for contradiction that $f(P) = f(P^0) = [a_l] \neq f(P'_1, P_{-1}) = f(P^1) = [a_r]$. Assume w.l.o.g that $P_1^L = P_1^{'L} \leq_L [a_l]$. There are three cases:

Case 1: Suppose $P_1^L = P_1^{'L} \leq_L [a_l] <_L [a_r]$. Voter 1 can deviate at profile P^1 from P'_1 to P_1 and be better-off at the profile P^0 by single-peakedness of $\succsim_i$. Since f is strategy-proof this is a contradiction. Case 2: Suppose $[a_r] <_L [a_l]$. Similar contradiction arises in the following two sub-cases: Case 2.1: $P_1^L \leq_L [a_r] <_L [a_l]$. Voter 1 can deviate at profile P^0 from P_1 to P'_1 and be better-off by single-peakedness of $\succsim_i$. Case 2.2: $[a_r] \leq_L P_1^L = P_1^{'L} \leq_L [a_l]$ with at least one equality holding strictly. Suppose P_1 is such $P_1^L = [a_t]$ and that all the alternatives to the left of a_t are preferred over alternatives to the right of a_{t+L-1} i.e. xP_1y for all $x < a_t$ and for all $a_{t+L-1} < y$. By single-peakedness of $\succsim_i$, this implies that $[a_r] \succ_i [a_l]$. Therefore, voter 1 will benefit from deviating at profile P^0 from P_1 to P'_1. Since f is *strategy-proof*, this is not possible. Therefore $f(P^0) = [a_l] = [a_r] = f(P^1)$. Similar arguments can be made for the case where $[a_l] \leq_L P_1^L = P_1^{'L}$. Therefore, $f(P) = f(P^0) = f(P^n) = f(P')$.

■

Therefore, the outcome of an interval efficient and *strategy-proof* I-SCC only depends on the top-L intervals of voters irrespective of the ordering of alternatives within that interval. In other words, the outcomes of any two profiles which have the same set of top-L alternatives are the same. Next, our main theorem provides a characterization of strategy-proof I-SCCs.

Theorem 1 *Suppose the extension $\succsim_i$ of preferences P_i for each voter $i \in N$ is responsive on intervals. An I-SCC, $f : \mathcal{S}^n \to \mathcal{I}_L$, is anonymous, strategy-proof and interval efficient if and only if it is a GMI rule.*

Proof: We prove necessity first. Anonymity follows from the definition of GMI rules. Strategy-proofness: Consider any voter $i \in N$ and a given profile $P \in \mathcal{S}^n$. There are two cases: (i) If $f(P) = P_i^L$ then there is no profitable deviation that leads to a strictly better outcome for i. (ii) If $f(P) <_L P_i^L$ then the only way to change the outcome is to change the median of the reported top-L intervals by all the voters $\{P_i^L\}_{i=1}^n$ and the fixed intervals $\{[\alpha_i]\}_{i=1}^{n-1}$. This can be done by reporting P'_i such that $P_i^{'L} <_L f(P)$. By single-peakedness over intervals and by the definition of GMI rule, $[f(P_i, P_{-i}) \leq_L f(P'_i, P_{-i}) <_L P_i^L] \implies [f(P_i, P_{-i}) \succsim_i f(P'_i, P_{-i})]$. Therefore no unilateral deviation can be strictly beneficial.

Interval Efficiency: Take any profile $P \in \mathcal{S}^n$ and suppose $[a_l] <_L f(P)$ for some interval $[a_l] \in \mathcal{I}_L$. Since $f(P) = med(P_1^L, ..., P_n^L, [\alpha_1], ..., [\alpha_{n-1}])$ all those voters i with $P_i^L >_L f(P)$ strictly prefer $f(P)$ over $[a_l]$ by single-peakedness over intervals. Therefore, any such interval $[a_l]$ cannot make all the voters strictly better-off compared to $f(P)$. Similar arguments can be made for the intervals on the right of $f(P)$. Therefore, GMI rules are interval-efficient.

We now prove sufficiency. Suppose an I-SCC, $f : \mathcal{S}^n \to \mathcal{I}_L$ is anonymous, strategy-proof and interval efficiency. We will show that it is a GMI rule. Note that interval efficiency of f implies *unanimity* i.e. if $P_i^L = P_j^L$ for all $i, j \in N$, then $f(P) = P_i^L$ since any other outcome would not be interval efficient (it can be improved upon by selecting P_i^L). By Proposition 2 the rule is top-L only. Our proof proceeds in steps. We first consider voter profiles where varying number of voters have peaks either at a_1 or a_m. We define the fixed intervals $([\alpha_1], ..., [\alpha_{n-1}])$ using the outcomes at those profiles. Finally, we show that the outcome for every profile is a GMI rule according to the given α's. We elaborate on the first step below.

Let $\underline{P}$: $a_1 \underline{P} a_2 \ldots \underline{P} a_m$ with the top-L interval as $\underline{P}^L = [a_1]$ and let $\overline{P}$: $a_m \overline{P} a_{m-1} \ldots \overline{P} a_1$ with the top-L interval as $\overline{P}^L = [a_{m-L+1}]$. Let $(\underline{P}^{n-k}, \overline{P}^k) \in \mathcal{S}^n$ be a profile where $n-k$ voters have the preference $\underline{P}$ and k voters have the preference $\overline{P}$ for any $k \in \{1, \ldots, n-1\}$. We define the fixed intervals for the GMI rule f^α as follows. Let $[\alpha_k] = f(\underline{P}^{n-k}, \overline{P}^k)$ for all $k \in \{1, \ldots, n-1\}$. Therefore, α_k is the outcome of the I-SCC f where k voters have the preference $\overline{P}$ and the remaining voters have the preference $\underline{P}$. These α's define a GMI rule f^α with $\{[\alpha_i]\}_{i=1}^{n-1}$. To show that the two rules coincide i.e. $f = f^\alpha$ we use the following Lemma.

Lemma 2 *For all $i \in \{1, \ldots, n-2\}$ we have $[\alpha_i] \leq_L [\alpha_{i+1}]$.*

Proof: Suppose a voter l changes preference from $\underline{P}$ to $\overline{P}$ such that the profile changes from $(\underline{P}^{n-k}, \overline{P}^k)$ to $(\underline{P}^{n-k-1}, \overline{P}^{k+1})$. Strategy-proofness requires that if the true preference of voter l is $\underline{P}$ then by deviating to $\overline{P}$ the outcome cannot be strictly preferred. Therefore, there are only two cases: (i) Suppose $f(\underline{P}^{n-k}, \overline{P}^k) \sim_l f(\underline{P}^{n-k-1}, \overline{P}^{k+1})$. This is only possible if $f(\underline{P}^{n-k}, \overline{P}^k) = f(\underline{P}^{n-k-1}, \overline{P}^{k+1})$ since $\succsim_i$ is anti-symmetric. Therefore, $f(\underline{P}^{n-k}, \overline{P}^k) = [\alpha_k] = f(\underline{P}^{n-k-1}, \overline{P}^{k+1}) = [\alpha_{k+1}]$. (ii) Suppose $f(\underline{P}^{n-k}, \overline{P}^k) \succ_l f(\underline{P}^{n-k-1}, \overline{P}^{k+1})$. By definition and single-peakedness over intervals, $[\alpha_i] <_L [\alpha_{i+1}]$. Hence, $[\alpha_1] \leq_L [\alpha_2] \cdots \leq_L [\alpha_{n-1}]$.

■

We now show that f is the GMI rule f^α with fixed intervals as $\alpha = \{[\alpha_i]\}_{i=1}^n$ i.e. for any $P \in \mathcal{S}^n$, $f(P) = f^\alpha(P)$. We apply induction on the number of voters (say, κ) who don't have their top as a_1 or a_m. Let κ be the induction variable such that $\kappa = |\{i : \tau(P_i) \notin \{a_1, a_m\}\}|$. We first argue for the base case where $\kappa = 0$ i.e. $P_i \in \{\underline{P}, \overline{P}\}$ for all $i \in N$. There are two cases here. Let $|\{i|\tau(P_i) = a_1\}| = n-k$ and $|\{i|\tau(P_i) = a_m\}| = k$. Case B1: Suppose $k \in \{0, n\}$. For all $i \in N$, either $\tau(P_i) = a_1$ ($P_i = \underline{P}$) or $\tau(P_i) = a_m$ ($P_i = \overline{P}$). By interval efficiency, $f(\underline{P}^n) = f^\alpha(\underline{P}^n) = \underline{P}^L = [a_1]$ and $f(\overline{P}^n) = f^\alpha(\overline{P}^n) = \overline{P}^L = [a_{m-L+1}]$ respectively. Case B2: Suppose $0 < k < n$. By Lemma 2, $[\alpha_i] \leq_L [\alpha_{i+1}]$ for all $i \in \{1, \ldots, n-2\}$. Therefore, by construction, $f(P) = f(\underline{P}^{n-k}, \overline{P}^k) = [\alpha_k] = med\{P_1^L, \ldots, P_n^L, [\alpha_1], \ldots, [\alpha_{n-1}]\} = f^\alpha(P)$. We have shown that when the top-ranked alternatives of voters is either a_1 or a_m, the outputs of f and f^α coincide, which is the base case ($\kappa = 0$). Using induction we will show that for any arbitrary profile the outputs of f and f^α coincide.

Induction Hypothesis: Suppose any $\kappa \in \{0, \ldots, n-1\}$ the two functions f and f^{α} coincide. We prove that for $\kappa + 1$, the outputs of f and f^{α} will also coincide. Consider any $P \in \mathcal{S}^n$ where $\kappa + 1$ voters do not have their top alternatives as a_1 or a_m. Pick any such voter j with top-L interval P_j^L. Assume for contradiction $[a_l] = f(P_j, P_{-j}) \neq f^{\alpha}(P_j, P_{-j}) = [b_l]$. We show that there is a unilateral deviation which will benefit some voter.

W.l.o.g. assume that $[a_l] <_L [b_l]$. There are three cases. Case 1: $P_j^L = [a_t] \leq_L [a_l] <_L [b_l]$. Consider the preference profile $P'' = (P_j'', P_{-j})$, where $P_j''^L = \underline{P}$. This falls under the case where κ voters do not have their top alternative as a_1 or a_m. Therefore, by the induction hypothesis, $f^{\alpha}(P'') = f(P'')$. By definition of GMI rule f^{α}, since j has moved her top-L interval on the same side of the previous outcome (which is a median), the outcome remains unchanged, i.e., $f^{\alpha}(P) = f^{\alpha}(P'')$. This implies that $f(P) = [a_l] <_L f(P'') = f^{\alpha}(P) = f^{\alpha}(P'') = [b_l]$. The following move for j at profile P'' is beneficial: $P_j'' \to P_j$ since due responsiveness/single-peakedness over intervals, $\left[\tau(\succsim_j'') = P_j''^L = \underline{P}^L \leq_L [a_l] <_L [b_l]\right] \implies \left[[a_l] \succ_j'' [b_l]\right] \implies \left[f(P) \succ_j'' f(P'')\right]$. Case 2: $P_j^L = [a_t]$, where, $a_t \in (a_l, b_l)$. There are two sub-cases: $\tau(P_j) > b_l$ or $\tau(P_j) \in [a_t, b_l]$. The following argument works for both the cases. Consider the preference P_j' and a preference extension $\succsim_j'$ such that $[b_l] \succ_j' [a_l]$. Note that responsiveness does not impose any restriction on how the intervals $[a_l]$ and $[b_l]$ are compared according to $\succsim_j$ since $[a_l] <_L \succsim_j' = [a_t] < [b_l]$. Therefore, such a preference $\succsim_j'$ exists in the domain $\mathcal{S}(\mathcal{I}_L)$. Now consider the move from P_j' to $P_j'' = \underline{P}$, where voter j gets better off since $f(P'') = [b_l] \succ_j' f(P) = [a_l]$. Case 3: Suppose $a_l <_L [b_l] <_L P_j^L = [a_t]$. Here by deviating to $P_j''^L = \overline{P}$ again the outcomes must coincide but since the median cannot change we have $f(P) = [a_l] < [b_l] = f^{\alpha}(P) = f^{\alpha}(P'') = f(P'')$. This move is beneficial since by single-peakedness over intervals we get $\left[[a_l] <_L [b_l] \leq_L [a_t]\right] \implies [[b_l] \succ_j [a_l]] \implies [f(P'') \succ_j f(P)]$.

Similar arguments can be made when $[a_l] >_L [b_l]$. Therefore, in both the cases, when $[a_l] \neq [b_l]$, there exists a profitable deviation for a given preference profile. This contradictions the assumption that f is not strategy proof. Therefore, $f(P) = [a_l] = f^{\alpha}(P) = [b_l]$ and the two rules coincide. This completes the induction argument, and the claim is true for all $\kappa \in \{1, 2, ..., n\}$. Therefore for all $P \in \mathcal{S}^n$, $f(P) = f^{\alpha}(P)$ with the fixed intervals $\alpha = (\alpha_1, ..., \alpha_{n-1})$ as defined above.

■

We provide an intuitive sketch of the proof of Theorem 1. Necessity is straightforward. *GMI rules* are *anonymous* since the rule is invariant to permutation of voters' preferences. *GMI rules* are *interval efficient* since they always pick an L-interval which lies between the left-most and right-most top interval of voters. This implies that any other interval which makes a voter strictly better-off will also make a voter strictly worse-off. We show that *GMI rules* are *strategy-proof.* Since *GMI rules* only take into account the top interval of voters, a voter i has to change her own top interval to change the outcome. Since the *GMI rule* picks the median of the top intervals and the fixed intervals, the only way to change the outcome is to 'report' the top interval on the other side of the interval outcome, $f(P)$ (the outcome under truthful reporting). As a result of the deviation, the outcome moves further away from the 'true' top-L interval of voter i. Since the extension $\succsim_i$ of P_i is single-peaked over intervals (Proposition 1) any such deviation will make voter i worse-off.

Sufficiency of the axioms is proved in multiple steps. By Proposition 1, a preference $\succsim_i$ over *intervals* is single-peaked according to $<_L$. This implies that intervals can be arranged from left to right and can be seen as 'alternatives' in the relevant interval-based single-peaked domain. We have shown that any I-SCC which is *strategy-proof* and *interval efficiency* must be top-L only (Proposition 2). This implies that only the top intervals determine the outcome of such I-SCCs.

The next part of proof involves identifying the fixed intervals, $[\alpha_1], \ldots, [\alpha_{n-1}]$. This is done by starting with a profile where all voters have peak interval starting at either $[a_1]$ or $[a_{m-L+1}]$, i.e., the extreme left and right intervals respectively. Once the fixed intervals for the GMI rule have been identified, an induction argument is applied on the number of voters who do not have the extreme left or right peak intervals. The induction argument then proceeds by contradiction. If the given I-SCC rule is not the GMI rule defined in the first step, then we can construct preference profiles where an individual can deviate to get a strictly better outcome. A deviation to any of the extreme preferences falls under the induction hypothesis case. These arguments then imply that the I-SCC which is strategy-proof, anonymous and interval efficient must be the GMI rule with the defined fixed intervals.

4.1 Necessity of responsiveness for the strategy-proofness of the GMI rule

In this section we show that responsiveness over intervals is a necessary condition for the GMI rule to be strategy-proof given the ordering over the set of alternatives. Using Proposition 1 we only need to show that preferences over intervals are single-peaked if the GMI rule is strategy-proof. To formally study this, we list the primitives of the model.

Suppose the alternatives are ordered $a_1 < a_2 < \ldots < a_m$ and every voter $i \in N$ has a single-peaked preference P_i over X. Voters have preferences $\succsim_i$ over $\mathcal{I}_L$ which are also complete and transitive. Let $\mathcal{P}(\mathcal{I}_L)$ be the set of all complete and transitive binary relations on $\mathcal{I}_L$. Let $\mathcal{S}(\mathcal{I}_L)$ be the set of all single-peaked preferences on $\mathcal{I}_L$. Let $\mathbb{D}$ denote any generic domain of complete and transitive preferences over $\mathcal{I}_L$.

A s.c.f. $f : \mathbb{D}^n \rightarrow \mathcal{I}_L$ is defined over n-tuples of $\pi = (\succsim_1, \succsim_2, \ldots, \succsim_n)$ and produces an alternative $f(\pi) \in X$. The GMI rule is defined as before since it only takes into account the top-L intervals of voters: an I-SCC, $f^\alpha : \mathbb{D}^n \rightarrow \mathcal{I}_L$, is a *GMI* rule if there exist $n-1$ fixed intervals $[\alpha_1], \ldots, [\alpha_{n-1}]$) such that for any $P \in \mathcal{S}^n$,

$$f^\alpha(P) = med(P_1^L, \ldots, P_n^L, [\alpha_1], \ldots, [\alpha_{n-1}]).$$

Theorem 2 *The set of single-peaked preferences over $\mathcal{I}_L$ is the largest set over which the class of GMI rules is strategy-proof, i.e., if $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$ then there exists a GMI rule f^α on $\mathbb{D} \subseteq \mathcal{P}(\mathcal{I}_L)$ with fixed intervals, $[\alpha_1], \ldots, [\alpha_{n-1}]$ and a profile $P \in \mathbb{D}^n$ on which f^α is not strategy-proof.*

Proof. Suppose $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$. We show that there exists a GMI rule, $f^\alpha : \mathbb{D}^n \to \mathcal{I}_L$ with fixed intervals, $[\alpha_1], \ldots, [\alpha_{n-1}]$ and a profile $P \in \mathbb{D}^n$ on which f^α is not strategy-proof. Since $a_1 < a_2 < \ldots < a_m$, $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$ implies that there exist intervals $A, B, C \in \mathcal{I}_L$ such that (i) $A < B < C$ or $C < B < A$ and (ii) for some voter $i^* \in N$, $\tau(\succsim_{i^*}) = A$, and $C \succ_{i^*} B$.

Suppose w.l.o.g that $A < B < C$ and $\tau(\succsim_{i^*}) = A$. We take the following GMI rule, f^α which has the following fixed intervals: $[\alpha_1] = [\alpha_2] = \ldots = [\alpha_{n-1}] = C$. Consider the following profile, $\pi \in \mathbb{D}^n$ with the top intervals as follows (since GMI only takes the top-L intervals into account): $\tau(\succsim_{i^*}) = A$ and $\tau(\succ_i) = B$ for all $i \in N \setminus \{i^*\}$. Note that $\pi \notin \mathcal{S}(\mathcal{I}_L)$ since $A \succ_{i^*} C \succ_{i^*} B$ even though $A < B < C$. By definition of the GMI rule,

$$
\begin{aligned}
f^\alpha(\pi) &= med(\tau(\succsim_1), \ldots, \tau(\succsim_{i^*-1}), \tau(\succsim_{i^*}), \tau(\succsim_{i^*+1}), \ldots, \tau(\succsim_n), [\alpha_1], \ldots, [\alpha_{n-1}]) \\
&= med(\underbrace{B, \ldots, B}_{i^*-1}, \underbrace{A}_{i^{*th} \text{ voter}}, \underbrace{B, \ldots, B}_{n-i^*}, \underbrace{C, \ldots, C}_{n-1}) = B \\
&= med(\underbrace{A}_{\text{voter } i^*}, \underbrace{B, \ldots, B}_{n-1}, \underbrace{C, \ldots, C}_{n-1}) = B
\end{aligned}
$$

However, if individual i^* deviates to $\succsim'_{i^*} = C$, then the outcome at $\pi' = (\succsim'_{i^*}, \succsim_{-i^*})$ is,

$$
f^\alpha(\pi') = med(\underbrace{C}_{i^*}, \underbrace{B, \ldots, B}_{n-1}, \underbrace{C, \ldots, C}_{n-1}) = C
$$

Since $C \succ_{i^*} B$, this move is beneficial for i^*. Therefore, we get a contradiction that f^α is not strategy-proof. This implies that $\mathbb{D} \subseteq \mathcal{S}(\mathcal{I}_L)$. By Proposition 1, since preferences P_i are single-peaked, and $\succsim_i$ are single-peaked, $\succsim_i$ is responsive. ■

Therefore, not only is responsiveness weaker than the max or min preference extension it is ‘somewhat’ necessary for the GMI rule to be strategy-proof. The condition under which the latter holds is that the preferences over alternatives be single-peaked. If the latter condition is violated, there is no guarantee that a single-peaked preference extension over intervals will imply single-peakedness over alternatives, thereby violating responsiveness over intervals (since Proposition 1 would no longer hold).[11]

[11] Preferences over alternatives need not be single-peaked for preferences over intervals to be single-peaked, e.g.: consider an ordered set of alternatives $a_1 < a_2 < a_3 < a_4 < a_5$ and $L = 3$. The preference $P_i = a_2 P_i a_4 P_i a_3 P_i a_1 P_i a_5$ is not single-peaked but the following responsive preference extension $\succsim_i$ is: $[a_2] \succsim_i [a_1]$ and $[a_2] \succsim_i [a_3]$.

5 Conclusion

We characterize *generalized median interval rules* on an extended single-peaked domain which satisfy responsiveness on intervals. It remains to be seen what the class of *strategy-proof* and *interval efficient* SCCs would be without responsiveness over intervals or if the aggregation rules picked non-intervals. This will depend on the nature of assumptions made on preference extensions to intervals or non-intervals.

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