Ashoka University
Economics Discussion Paper 111

Fair allocation with (semi-single-peaked) preferences over location and quantity

April 2024

Mihir Bhattacharya, Ashoka University
Ojasvi Khare, Economics and Planning Unit, Indian Statistical Institute

https://ashoka.edu.in/economics-discussionpapers

Plot No. 2, Rajiv Gandhi Education City,
National Capital Region,
P.O.Rai, Sonepat, Haryana 131029

Fair allocation with (semi-single-peaked) preferences over location and quantity*

Mihir Bhattacharya† Ojasvi Khare‡

Abstract

We consider the problem of dividing and allocating a perfectly divisible heterogeneous good where agents have a preference for location and quantity. We assume that preferences are single-peaked in quantity, i.e., *semi-single-peaked* which can be represented by continuous indifference curves (ICs). We show existence of envy-free and Pareto efficient allocation rules, and characterize the set of all such rules using the notion of a *balanced* IC. We define the *balanced-curve allocation (BCA)* which uses the region between the two balanced ICs to obtain feasible allocations. We show that an allocation rule is *envy-free* and *Pareto efficient* if and only if it is in the set specified by the *BCA* rule. We show that there is no *strategy-proof*, *envy-free* and *Pareto efficient* allocation rule. We provide some insights into the problem when there are more than 2 agents.

JEL classification: D01, D70, D71

Keywords: single-peaked, location, quantity, allocation, envy-free.

*We would like to thank Debasis Mishra, Bhaskar Dutta and Arunava Sen for comments and suggestions.

†Department of Economics, Ashoka University, Rajiv Gandhi Education City, Rai, Sonipat, NCR, Haryana, 131029, India. Email: mihir.bhattacharya@ashoka.edu.in

‡Economics and Planning Unit, Indian Statistical Institute, Delhi Center, 7, S. J. S. Sansanwal Marg, New Delhi 110016. Email: ojasvi19r@isid.ac.in

1 Introduction

There are many settings where a contiguous set or interval of a heterogeneous and perfectly divisible resource is to be allocated among multiple interested agents. Agents may have preference for the location and the quantity of the interval depending on where it is located. Some examples include allocating a piece of land for the construction of a facility or a time interval which has to be allocated to an advertisement firm. We propose a new restricted preference domain where the preference for quantity is single-peaked. We characterize *envy-free* and Pareto efficient allocations in this setting.

The idea of envy-freeness was introduced in Foley (1966) and further studied in Varian (1973) and Thomson and Varian (1984). Envy-freeness is one of the central axioms of fairness and economic equity in the allocation literature (Thomson and Varian (1984)).[1] The condition requires that for any given allocation problem, every agent must prefer her own allocation to the allocation of any of the other agent. In spite of the appeal of the axiom, characterizing envy-free allocations in such models is extremely hard and no finite bounded algorithms exist (Stromquist (2008)). We evade the problem of complexity and existence in such models by considering an ordinal model where the preference for quantity is single-peaked.

We consider an allocation model where agents have preferences over intervals represented by the tuple (x, q) where x denotes the starting point or the 'location' of the interval and q denotes the length or the quantity of the interval. Therefore, each point (x, q) corresponds to an interval, $[x, x+q]$, which is a closed connected subset of the unit interval $[0, 1]$. We assume that agents have continuous preferences which are represented by orderings.[2] We assume that for any agent i and any given quantity $q' \in [0, 1]$, an indifference curve function $IC_{i,q'}$ represents the quantity $IC_{i,q'}(x)$ for any location x such that the bundle $(x, q = IC_{i,q'}(x))$ is indifferent to the allocation $(0, q')$. Figure 1 shows the domain of preferences and illustrates an IC passing through $(x, IC_{i,q}(x)))$. These ICs cut across the domain and intersect the two axes: one at $x = 0$ and the other at $x + q = 1$.

We assume that each agent has a 'top' IC which connects the set of most preferred allocations. We define single-peakedness in quantity as follows: for each location x in the domain of an $IC_{i,q}(x)$, the bundles further away from the any (x, q) on the top IC

[1] Varian (1973) defines an allocation to be *fair* if it is Pareto efficient and equitable (i.e. envy-free), this is inspired by the idea that fairness of allocation may be contributed by agents judgement of their allocation, others' allocation and the comparison thereof.

[2] A binary relation R is called a *preference ordering* if it is (i) complete: xRy or yRx for all x, y and (ii) transitive: xRy and yRz implies xRz for all x, y, z. A preference, $\succ$ on X is continuous on X if and only if for any $x \in X$, the lower and upper contour sets, $\{y : x \succsim y\}$ and $\{y : y \succsim x\}$ respectively, of x are closed.

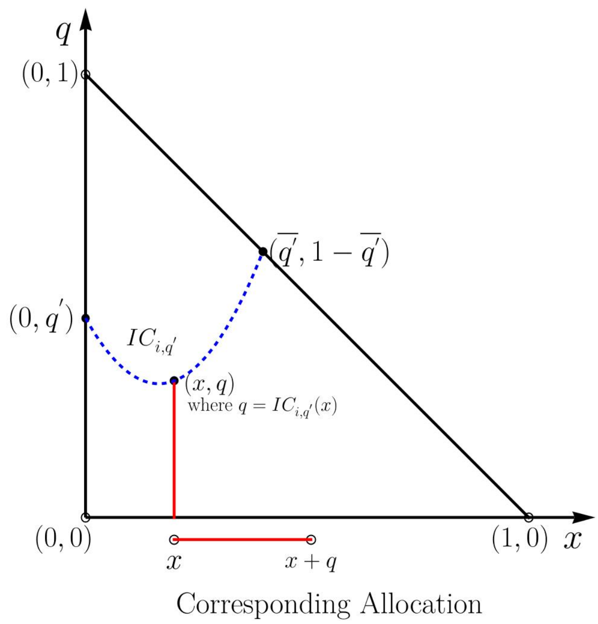

Figure 1: The set of alternatives X is the set of all the points in the triangle with vertices $\{(0,0),(0,1),(1,0)\}$

are worse. In other words, for any (x, q) on the top IC, since q is the *most* preferred quantity at location x, for any $q \leq q' < q''$ or $q'' < q' \leq q$ the bundle (x, q') is strictly better than the bundle (x, q''). An implication of this assumption and transitivity of preferences is that ICs 'further away' from the top IC connect bundles which are strictly worse than the bundles on ICs which are closer to the top IC.

An implication of single-peakedness in quantity is that agents may not always want more quantity. Semi-single-peaked preferences are natural to assume in the following settings:

- Plot of land: an interval of land has to be allocated to each agent. Some agents may prefer to have more (or less) quantity depending on the location due to additional benefits (or cost) at those locations. The marginal gain from a bigger piece of land may be less than the additional cost.
- Advertisement slots: agents need to have advertise their product in different slots over the unit time interval. Agents may want slots at different times when their 'target' audience is attentive. Smaller slots may not be sufficient and larger slots may not be worth the additional cost.

Our model is two-dimensional in the sense that location and quantity are the two dimensions where the latter depends on the first dimension. Our model, therefore, can be seen as a generalization of the one-dimensional single-peaked allocation models where we allow for agents to have preferences over location and quantity. However, a crucial significant difference between our model and the standard notion of single-peakedness is that preferences over location are not single-peaked. Therefore, we use the term *semi-single peaked* to denote these preferences. Table 1 provides a list of essential differences between the preferences defined in our model: *semi-single-peakedness*, classical monotonic preferences and single-peaked preferences.[3]

Single-peaked preferences	Classical preferences	Semi-single-peaked
Convex	Convex	Non-convex
Not monotone	Monotone	Not monotone
Satiation	Non-satiation	Satiation
Continuous	Continuous	Continuous
Allocations further away from the peak are worse	More is always better	Allocations further away from the *peak quantity at a given location are worse*

Table 1: Comparison of different preferences

In this paper we characterize *envy-free* and Pareto efficient allocations for two agents and provide some insights for the case with more than two agents. Envy-freeness requires that in any allocation of goods each agent should weakly prefer her own allocation to any of the others'. When agents have identical preferences, envy-freeness therefore requires agents to be given allocations on the same IC.

A preference profile that is frequently used in our paper is where both the agents have identical preferences and prefer to have as much quantity as possible for any given location, i.e., preferences are *monotonic* in quantity. This corresponds to the case when the top IC consists of a single point $(0,1)$. Envy-freeness requires that both agents must receive an allocation on the same IC. However, if they receive two allocations (x,q) and (x',q'), Pareto efficiency requires that $q+q'=1$. Moreover, since intervals cannot intersect (except at an end-point) we must have $x'+q'=q+q'=1$.[4]

We provide results for 2 agents. To provide our main result, we first show that when preferences of the agents are such that their top ICs lie in a 'high enough' region, both agents must receive allocations in the *balanced curve region* which is the region between the two balanced ICs. An IC is *balanced* if it cuts the two axes $x=0$ and $x+q=1$ at the points $(0,q_i^f)$ and $(q_i^f,1-q_i^f)$ respectively. In other words, if we were to give an allocation to either agent on the same IC of some agent and also ensure

[3] The preference relation $\succsim$ satisfies convexity if $x \succsim y$ and $\alpha \in (0,1)$ then $\alpha x + (1-\alpha)y \succsim y$.

[4] An assumption on the slope of the ICs at $(0,q)$ for any $q \in (0,1)$ ensures that agents receive allocation on the two axes of the domain i.e. either at $x=0$ or on the line $x+q=1$. The allocation allows for intersection of the two allocated intervals at one end point. However, since a singleton set is a set of measure zero, the representative utility functions would not distinguish between the intervals, $[x,x+q]$ and $[x,x+q)$.

that we give away the whole resource, it would have to be on the balanced IC of that agent. We prove the existence of a balanced IC for any given set of preferences using a fixed point argument. Moreover, the balanced IC is unique, which can be proved using the following argument. If there is more than one balanced IC for a given set of preferences, then those ICs would have to intersect. However, this would contradict transitivity of preferences.

We describe the *balanced curve algorithm (BCA)* which provides the full set of envy-free and Pareto efficient allocations. When agents have a 'high enough' top ICs, in most cases, both agents receive allocations in the balanced IC region: the region of allocations between the two balanced ICs. An agent receives allocations on the left axis when her balanced IC cuts the balanced IC of the other agent from below on the $x = 0$ axis. An assumption on the slope of ICs (slope greater than or equal to -1) on the latter axis ensures that under the above condition, agents will always prefer to receive allocations on the axis $x = 0$ rather than at some point in the interior when the same quantity is available.[5] Our first observation shows that when agents have identical preferences and prefer higher quantity, then any envy-free and Pareto efficient allocation must assign them allocations on their balanced IC (here both have the same balanced IC since their preferences are same).

The BCA allocates bundles on the top ICs of the two agents whenever that is feasible. However, there are multiple cases where this is not possible. There are broadly three cases: in the first case, both agents prefer to have as much quantity as possible. In this case, both agents receive the balanced IC region explained above. The second case is where one of the agents does not want 'too much' quantity, i.e., her top IC lies somewhere below the balanced IC region. In this case, the set of Pareto efficient and envy-free allocations include bundles on her top IC and all other bundles up to the point where she is indifferent between the bundle given to the other agent on the opposite axis. This ensures that there is no envy. Similarly, there are Pareto indifferent allocations where agents get allocations on opposite axes. There are multiple sub-cases where the top IC of an agent may intersect the either of the two axes in the balanced IC region. In such cases, envy-free and Pareto efficient allocations may be given up to the point of intersection with the top IC. This is due to the fact that it is not Pareto efficient to give more quantity than the peak quantity to any agent at a given location. The final case is where both agents do not want a 'lot of' quantity and are given any of the feasible allocations on their top ICs on either axis.

Our main insight from the first result is that the class of envy-free and Pareto efficient allocation rules is large and multi-valued. The algorithm is fairly simple to calculate

[5] Observation 2 shows that this assumption is necessary for existence of envy-free and Pareto efficient allocations.

in principle: one can 'travel' between the top ICs of both the agents, starting with the allocations of one agent on the top IC on either side of the two axes and prevent envy on the way. This also ensures Pareto efficiency since the allocations lie between the region of the top ICs.

Given that the set of envy-free and Pareto efficient set of rules is large, a natural question arises: are some of these rules strategy-proof, i.e., do they ensure truthful revelation of preferences? We find that for 2 agents, there are no strategy-proof, envy-free and Pareto efficient rules. In other words, there exist profiles where an agent can beneficially deviate by misreporting her preferences. This is not surprising given the richness of the domain and due to semi-single peakedness of preferences.

For more than three agents, we provide some arguments for the existence of a k-balanced IC which ensures that any k agents can be given allocations on the same IC without wastage (excess or deficit). We provide a necessary condition for an envy-free and Pareto efficient allocation. We indicate the possibility of generalizing the BCA allocation from 2 to any k number of agents.

There are many papers studying envy-free and group envy-free allocations in the Walrasian equilibrium setting. However, all these models assume canonical preferences which are convex and monotonic. The preferences in our model are not monotonic with respect to quantity and hence, their results do not apply to our model. Thomson (1994) studies resource monotonic and envy-free allocations in the single-peaked setting where the only dimension is quantity, while Sprumont (1991) consider a similar setting to characterize the uniform rule as the only strategy-proof, anonymous and efficient allocation rule. Both these works characterize the uniform rule which do not have a direct analogous version in our setting. The closest version of this rule in our model would be the allocation where each agent gets an n-th fraction of the quantity. However, the location of such intervals would be different and agents may rank different pieces differently. Due to these reasons our model does not reduce to the one-dimensional single-peaked model when location 'does not matter' (when the ICs have zero slope everywhere).[6]

Thomson (1994) characterizes the uniform rule as the only rule that is Pareto efficient, envy-free and depends on peak quantity. Our BCA allocation is similar only in spirit to the uniform allocation due to reasons mentioned above. Sprumont (1991) studies the division problem when there is a fixed quantity of a perfectly divisible good to be allocated among agents and each agent has a single-peaked preference over the quantity allocations. Our model can be seen as a generalization of the one-dimensional single-peaked model where location and quantity both matter.

[6]Thomson (2011) and Thomson (2016) provide excellent surveys of the related literature on envy-free allocations.

Our approach in this paper is fairly general and the preferences in our model cannot be represented by a single-variable valuation function over a one-dimensional space as in the cake-cutting literature (Procaccia (2016)). This is due to the fact that agents may have varying preferences over different quantities given a fixed location of the interval. Therefore, for continuous valuation functions, a strict subset of the interval which has the highest valuation will always have a strictly smaller valuation if the valuation of the bigger interval is positive. Since we do not assume this property, a single valuation function would not be able to characterize semi-single-peaked preferences.[7] The analogous cardinal version of our model would require a valuation function $f_y(x)$ for every given location y in the unit interval. However, computing envy-free allocations would be even more challenging.

Brams et al. (2013) and Brams et al. (1995) highlight the difficulties in finding a fair allocation of the cake when the number of agents is large. The latter work provides moving-knife algorithms to obtain envy-free allocations. Lindner and Rothe (2015) notes that "*...despite intense efforts over decades, up to this date no one has succeeded in finding a finite bounded cake cutting protocol that guarantees envy-freeness for any number of players...*". Similarly, Stromquist (2008) provides an impossibility result for envy-free cake divisions by finite protocols. Many papers in the cake-cutting literature consider normalized valuation functions to get positive results (see Chen et al. (2013), Aumann and Dombb (2015) and Caragiannis et al. (2012)). The approach taken in our paper is different, due to the ordinal nature of preferences. Moreover, the property of single-peakedness in quantity allows us to obtain positive results in this setting.

Bogomolnaia and Moulin (2023) studies the divide-and-choose and moving knife rules and provide conditions for minimum guarantees when preferences are represented by continuous utility functions (but may not be monotone or convex). In our model, when agents have monotonic preference over quantity, the BCA guarantees at least the worst allocation in the balanced IC region. This property may also hold for more than 3 agents. We hope that future work will provide further insights into this. Aziz and Mackenzie (2016) provides envy-free and bounded algorithms for cake-cutting for any number of agents. However, their procedure requires $n^{n^{n^{n^{n^n}}}}$ queries. Therefore, an additional advantage of our model is that the envy-free and Pareto efficient rules are well-defined and fully characterized.

The paper is organized as follows. Section 2 discuss the model, preferences and the assumptions which characterize semi-single peaked preferences. Section 3 describes the axioms and Section 4 provides the results for 2 agents. Section 5 provides some

[7] More specifically if $I = [a, b] \subset [0, 1]$ is the interval with the highest valuation, which is given by $V(I) = \int_a^b f(x)dx$ for a continuous valuation function $f(x)$, then for any interval $I' \subset I$, $V(I') \leq V(I)$. However, the preferences in our model allow for the set $I' = [a', b'] \subset I$ to be higher valued if $a' \neq a$.

observations for more than 2 agents and Section 6 provides the conclusion.

2 Model

A heterogeneous and perfectly divisible resource is distributed on an interval $[0,1]$ and has to be divided among n agents. The set of agents is $N = \{1, 2, \dots, n\}$. The interval has to be divided such that each agent receives a closed interval from $[0,1]$ which must be disjoint except at a single point from the interval received by another agent.[8] Formally an *allocation* of agent i will be denoted by $a_i = (x_i, q_i) \equiv [x_i, x_i + q_i]$ where $x_i \in [0,1]$ is the starting point of the interval and $q_i \in [0, 1 - x_i]$ it's length (or the quantity). The unit interval can be seen as a time interval or land which has to be fully distributed to the agents. An *allocation*, $a = \{(x_i, q_i)\}_{i \in N}$, is said to be *feasible* if (i) for any $i \in N$, $q_i > 0$, $\sum_i q_i \leq 1$ and (ii) for any pair of distinct agents i and j, $x_i + q_i \leq x_j$ or $x_j + q_j \leq x_i$. Therefore, condition (i) ensures that the resource is allocated amongst the n agents with each receiving a positive quantity while the second condition ensures that the intervals do not intersect except at a single point, which are sets of measure zero. Note that we allow for *free disposal* i.e. we do not need to allocate the full resource to all the agents.

Another way to visualize the allocations is to arrange the agents who have received allocations from left to right, i.e., let $i^* \in \{1, ..., n\}$ be such that $x_{1^*} < x_{2^*} < ... < x_{n^*}$. Then it must be the case that $x_{2^*} = x_{1^*} + q_{1^*}$, $x_{3^*} = x_{2^*} + q_{2^*}$, ..., $x_{n^*} = x_{n^*-1} + q_{n^*-1}$. For example, for three agents, the feasible allocation: $a = (a_1, a_2, a_3)$ such that $a_1 = (0, \frac{1}{2})$, $a_2 = (\frac{1}{2}, \frac{3}{10})$ and $a_3 = (\frac{8}{10}, \frac{2}{10})$ indicates that agent 1 has been given the interval $[0, \frac{1}{2}]$, agent 2 has been given the interval $[\frac{1}{2}, \frac{8}{10}]$ and agent 3 has been given the interval $[\frac{8}{10}, 1]$. The set of all feasible allocations for n agents shall be denoted by $\mathcal{A}$.[9] An allocation $\{(x_i, q_i)\}_{i \in N}$ is said to be a *no wastage* allocation if $\sum_i q_i = 1$.

The allocation of any agent can be visualised as a point in the right-angled isosceles triangle with vertices at $(0,0)$, $(0,1)$ and $(1,0)$. For any $x, q \in [0,1]$ such that $x + q < 1$, we denote a right-angled isosceles triangle with vertices at $(x, 0)$, $(x+q, 0)$ and (x, q) by T_x^q.

Therefore, the set of alternatives can be denoted as $X = \{(x, q) |\ 0 \leq x, 0 \leq q, x + q \leq 1\}$ or $X = T_0^1$. Next, we define the set of preferences in this domain through a

[8] This is a standard assumption in the fair division literature (Procaccia (2016), Thomson (2011)).

[9] Note that this set cannot be a cross-product of any set since one's allocation depends on the other.

sequence of assumptions.

Semi-single-peaked preferences. We define a type of preference over X. These are motivated by the interpretation of single-peaked preferences defined over a line (see Sprumont (1991) for example). In this paper we will consider preferences which will be single-peaked with respect to quantity for *a given* location $x \in [0, 1]$. Our first assumption is a standard one.

Assumption 0: We assume that preference $\succsim_i$ of each agent $i \in N$ is *complete*, *transitive* and *continuous* which can be represented by indifference curve functions, $IC_{i,q} : [0, \bar{q}] \to [0, 1]$ for any $q \in [0, 1]$ where $IC_{i,q}(\bar{q}) = 1 - \bar{q}$. We will denote the preferences by $\succsim$ where $\succ$ denotes the asymmetric part of the relation and $\sim$ denotes indifference.

We note that by Debreu et al. (1954) and Debreu (1959) these preferences can be represented by continuous utility functions, $u : T_0^1 \to \mathbb{R}$. Assumption 0 requires that ICs are mappings from locations to quantities within T_0^1.

For every $x \in [0, \bar{q}]$, the value $IC_{i,q}(x)$ is the quantity such that the bundle $(0, q)$ is indifferent to the allocation $(x, IC_{i,q}(x))$ i.e. $(0, q) \sim_i (x, IC_{i,q}(x))$ for all $x \in [0, \bar{q}]$. The bundle $(\bar{q}, 1-\bar{q})$ can also be seen as the right-most bundle on $IC_{i,q}$ which intersects the line $x + q = 1$ at $(\bar{q}, 1 - \bar{q})$.

Another interpretation of the function $IC_{i,q}(x)$ is that it indicates the quantity required at x for the bundles $(x, IC_{i,q}(x))$ to be indifferent to each other. Next, we impose some properties on the functional form of $IC_{i,q}$ for every $i \in N$ and $q \in [0, 1]$.

Assumption 1: **Existence of a top IC.** We assume that for every agent $i \in N$ there is a *top indifference function*, $IC_{i,q_i^\tau} : (0, \bar{q}_i^\tau) \to [0, 1]$ for some $q_i^\tau \in (0, 1]$ such that $(x, IC_{i,q_i^\tau}(x)) \succsim_i (x', q')$ for all $x \in [0, \bar{q}_i^\tau]$ and for all $(x', q') \in T_0^1$. We shall denote $IC_{i,q_i^\tau}(x)$ by IC_i^τ. Note that the top IC, as illustrated in Figure 2, starts at $(0, q_i^\tau)$ on the $x = 0$ axis, and intersects the $x + q = 1$ axis at the point $(\bar{q}_i^\tau, 1 - \bar{q}_i^\tau)$.

We assume that all agents want non-zero quantities i.e. $q_i^\tau \in (0, 1]$ for all $i \in N$. We denote the set of top-ranked allocations as $\tau(\succsim_i) \equiv \{(x, q) | (x, q) \sim_i (0, q_i^\tau)\}$. Note that if $q^\tau = 1$ for some agent $i \in N$, then $\tau(\succsim_i) = \{(0, 1)\}$ i.e. her top IC is just the allocation $(0, 1)$. Note that any agent $i \in N$, q_i^τ and $\bar{q}_i^\tau$ are such that $(0, q_i^\tau), (\bar{q}_i^\tau, 1 - \bar{q}_i^\tau) \in \tau(\succsim_i)$ which are the two allocations on IC_{i,q_i^τ} which cut the $x = 0$ axis and the $x + q = 1$ line respectively.

Assumption 2: **Lowest at zero quantity.** We will assume that for any $i \in N$ and for all x, $0 < x \leq 1$, $(0, 0) \sim_i (x, 0)$ i.e. allocations of zero length are indifferent to each other and that for all $q > 0$, $(0, q) \succ_i (0, 0)$, all allocations with positive quantity are better than no allocation.

Assumption 3: **Single-peakedness in quantity.** For any $i \in N$ and all q', q'' such that (i) $q_i^\tau \leq q' < q''$ or (ii) $q'' < q' \leq q_i^\tau$ implies that $(0, q') \succ_i (0, q'')$.

Transitivity of the preferences imply that the ICs do not intersect (Mas-Colell et al. (1995)). Therefore, an implication of Assumption 3 and transitivity is that for all $x \in [0,1]$ such that (x, q') and (x, q'') are in T_0^1 (i) $q_i^\tau \leq q' < q''$ or (ii) $q'' < q' \leq q_i^\tau$ implies that $(x, q') \succ_i (x, q'')$.

As a result of transitivity and single-peakedness in quantity it will be useful to call an $IC_{i,q}$ 'closer' to the top IC, IC_{i,q_i^τ} than another $IC_{i,q'}$ on the same side of the top IC if for some $a \in [0,1]$, (a, q_1) lies on IC_{i,q_i^τ}, (a, q_2) lies on $IC_{i,q}$, (a, q_3) lies on $IC_{i,q'}$ and either (i) $q_1 \leq q_2 < q_3$ or (ii) $q_3 < q_2 \leq q_1$. Moreover, transitivity also implies that for any (x_1, q_1) on $IC_{i,q}$ and (x_2, q_2) on $IC_{i,q'}$ it will be the case that $(x_1, q_1) \succ_i (x_2, q_2)$ if $IC_{i,q}$ is 'closer' to the top IC than $IC_{i,q'}$ in the manner described above.

Semi-single-peaked preference domain: The set of all semi-single-peaked preferences satisfying Assumptions 0 to 3 over $X = T_0^1$ is said to be the semi-single-peaked preference domain. We denote such a domain as $\mathcal{D}$. Therefore the relevant preference profiles are such that $\mathbf{P} = (\succsim_1, \succsim_2, ..., \succsim_n) \in \mathcal{D}^n$.

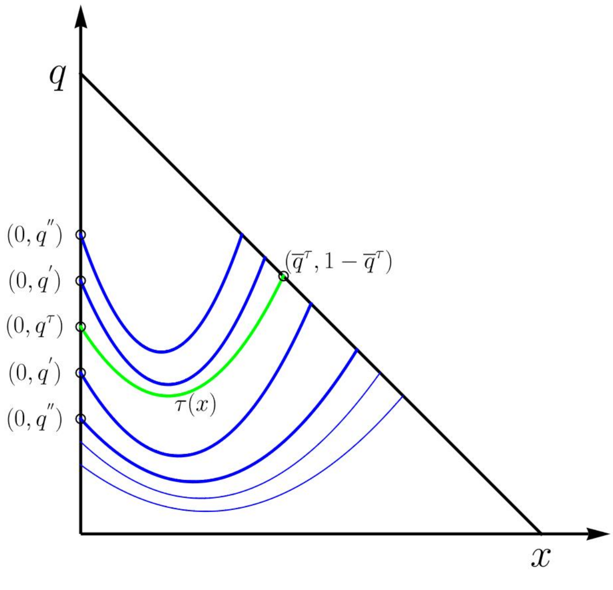

Figure 2: An example of semi-single-peaked preferences

A *allocation rule* is a function, $f : \mathcal{D}^n \to \mathcal{A}$, that takes in a preference profile $\mathbf{P} \in \mathcal{D}^n$ and produces a feasible allocation $f(P) = (f_1(\mathbf{P}), f_2(\mathbf{P}), \ldots, f_n(\mathbf{P})) \in \mathcal{A}$, where $f_i(\mathbf{P}) \in T_0^1$ is the allocation of agent $i \in N$. We now present the axioms.

3 Axioms

The first axiom is the standard notion of efficiency: an allocation is Pareto efficient if no agent can be made strictly better-off without making another agent strictly worse-off.

Pareto Efficiency. An allocation rule, $f : \mathcal{D}^n \to \mathcal{A}$, is *Pareto efficient* if for any preference profile $\mathbf{P} \in \mathcal{D}^n$ there does not exist another feasible allocation $\{(x_i, q_i)\}_{i \in N} \in \mathcal{A}$ s.t. $(x_i, q_i) \succsim_i f_i(\mathbf{P})$ for all $i \in N$ and $(x_j, q_j) \succ_j f_j(\mathbf{P})$ for some $j \in N$.

In other words, an allocation is Pareto efficient if for any other allocation whenever an agent is strictly better-off, there is another agent who is strictly worse-off.

Anonymity. An allocation rule, $f : \mathcal{D}^n \to \mathcal{A}$, is *anonymous* if for every preference profile $\mathbf{P} \in \mathcal{D}^n$, and for each permutation σ of N, and for all $i \in N$, $f_{\sigma(i)}(\mathbf{P}) = f_i(\mathbf{P}_\sigma)$, where $\mathbf{P}_\sigma = (\succsim_{\sigma(1)}, ..., \succsim_{\sigma(n)})$. In other words, if the preferences of agents are permuted, then the allocations should also be permuted in the same way. Another version of anonymity, which is more appropriate in our setting is the following.

Anonymity*. An allocation rule, $f : \mathcal{D}^n \to \mathcal{A}$, is *anonymous**, if for every preference profile $\mathbf{P} \in \mathcal{D}^n$, and for each permutation σ of N, for all $i \in N$,

$$f_{\sigma(i)}(\mathbf{P}) \sim_{\sigma(i)} f_i(\mathbf{P}_\sigma)$$

where $\mathbf{P}_\sigma = (\succsim_{\sigma(1)}, ..., \succsim_{\sigma(n)})$.

According to Anonymity*, an allocation rule is anonymous* if for every permutation of a preference profile, the allocation rule ensures that individuals get an allocation which is indifferent to the preference of the permuted agent. Our main axiom, is a fairness condition, that of envy-freeness.

Envy-Free. An allocation rule, $f : \mathcal{D}^n \to \mathcal{A}$, is said to be *envy-free* if for all $\mathbf{P} \in \mathcal{D}^n$, for all $i, j \in N$, $f_i(\mathbf{P}) \succsim_i f_j(\mathbf{P})$.

An allocation rule is envy-free if every agent prefers her own allocation to any other agent's allocation. Our final axiom, strategy-proofness, is an inter-profile condition.

Strategy-proof. An allocation rule, $f : \mathcal{D}^n \to \mathcal{A}$, is said to be *strategy-proof* if for any $i \in N$ and for any $(\succsim_i, \succsim_{-i}) \in \mathcal{D}^n$,

$$f_i(\succsim_i, \succsim_{-i}) \succsim_i f_i(\succsim_i', \succsim_{-i}) \quad \text{for all } \succsim_i' \in \mathcal{D}.$$

An allocation is strategy-proof if it does not provide any incentive for individual agents to benefit strictly by misreporting their preference.

4 Results

In this section we present our results for 2 agents first. Our first result shows that standard anonymity is not an ideal property in this domain and also applies for more than 2 agents.

Proposition 1 *There is no allocation rule that is anonymous.*

Proof. Let $\succsim_1=\succsim_2=\succsim$ such that all ICs of $\succsim$ have zero slope and $\tau(\succsim) = \{(0,1)\}$. Pick the non-trivial permutation for $n = 2$, $\sigma(1) = 2$ and $\sigma(2) = 1$. Let $\mathbf{P} = (\succsim_1, \succsim_2)$, which implies that $\mathbf{P}_\sigma = (\succsim_2, \succsim_1)$. Anonymity requires that $f_i(\mathbf{P}) = f_j(\mathbf{P}_\sigma)$, where $i, j \in \{1,2\}$ and $i \neq j$. W.l.o.g. assume that $a_1 = f_1(\mathbf{P}) = (x_1, q_1)$ and $a_2 = f_2(\mathbf{P}) = (x_2, q_2)$ and that $a = (a_1, a_2) \in \mathcal{A}$. But note that $f_1(\succsim_1, \succsim_2) = f_1(\succsim_2, \succsim_1) = f_1(\succsim, \succsim)$. Similarly, $f_2(\succsim_1, \succsim_2) = f_2(\succsim_2, \succsim_1)$. Since $a_1 \neq a_2$ whenever $a = (a_1, a_2) \in \mathcal{A}$, $f_1(\succsim_1, \succsim_2) = a_1 \neq f_2(\succsim_2, \succsim_1) = a_2$. Therefore, anonymity is violated. ■

Next, we study envy-free and Pareto efficient allocation functions. It is easy to verify that for any $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}^n$ such that $\succsim_1=\succsim_2=\succsim$ then $f_1(\mathbf{P}) \sim_i f_2(\mathbf{P})$ for $i \in \{1,2\}$ i.e. both the agents must be indifferent to each other's allocation. If this is not the case then $f_1(\mathbf{P}) \succ_1 f_2(\mathbf{P})$ will imply that $f_1(\mathbf{P}) \succ_2 f_2(\mathbf{P})$ due to identical preferences. However, this is not envy-free. We provide an example of envy-free and Pareto efficient allocation.

Example 1 *Suppose* $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}^n$ *such that* $\succsim_1=\succsim_2=\succsim$ *and both prefer to have the whole interval i.e. their top IC is* $\tau(\succsim_i) = \{(0,1)\}$. *Pareto efficiency requires that* $f(\mathbf{P}) = (a_1, a_2) = ((x_1, q_1), (x_2, q_2))$ *such that* $q_1 + q_2 = 1$. *The above observations imply that under identical preferences* $(x_1, q_1) \sim_i (x_2, q_2)$ *for each* $i \in \{1,2\}$. *This implies that these allocations need to be on the same IC of the agents which we call balanced ICs. We define them formally.*

Balanced IC: For any $i \in N$ the *balanced IC* of agent i is an IC of the form IC_i^f if there exists a $q_i^f \in (0,1)$ such that $(0, q_i^f) \sim_i (q_i^f, 1 - q_i^f)$. In other words, a balanced IC of agent i consists of two allocations $(0, q_i^f)$ and $(q_i^f, 1 - q_i^f)$ which lie on the same IC. Our next proposition proves that there always exists a balanced IC for every agent $i \in N$ and that it is unique.

Proposition 2 *For any* $i \in N$ *with preference* $\succsim_i \in \mathcal{D}$, *there exists an IC,* $IC_{i,q_i^f} : [0, q_i^f] \to [0,1]$ *for some* $q_i^f \in (0,1)$ *such that* $(0, q_i^f) \sim_i (q_i^f, 1 - q_i^f)$. *Moreover,* q_i^f *is unique for every preference* $\succsim_i \in \mathcal{D}$. *We will denote* IC_{i,q_i^f} *by* IC_i^f.

Proof. We prove by contradiction. Suppose this is not the case. Then for each $q \in (0,1)$, either (i) the $IC_{i,q}$ IC cuts the $x + q = 1$ line above the point $(q, 1-q)$ or

(ii) the $IC_{i,q}$ IC cuts the $x + q = 1$ line below the point $(q, 1-q)$. Consider the IC at $q = \frac{1}{2}$, $IC_{i,\frac{1}{2}}$ which is the IC passing through the points $(0, \frac{1}{2})$ and $(\gamma, 1-\gamma)$ for some $0 < \gamma < 1$. We provide arguments for different cases. Note that if $\gamma = \frac{1}{2}$, then $q_i^f = \frac{1}{2}$ and our claim is true.

Case (i): $\gamma < \frac{1}{2}$, then for some $\alpha \in (0, \frac{1}{2})$, we will have $IC_{i,\frac{1}{2}-\alpha}(\frac{1}{2}) = \frac{1}{2}$. Now consider ICs between $IC_{i,\frac{1}{2}}(x)$ and $IC_{i,\frac{1}{2}-\alpha}(x)$ s.t. $IC_{i,\frac{1}{2}-\varepsilon}(\frac{1}{2} - \delta_1(\varepsilon)) = \frac{1}{2} + \delta_1(\varepsilon)$ for some function $\delta_1 : [0, \alpha] \to [0, \frac{1}{2} - \gamma]$ such that, $\big(0, \frac{1}{2} - \varepsilon\big) \sim_i \big(\frac{1}{2} - \delta_1(\varepsilon), \frac{1}{2} + \delta_1(\varepsilon)\big)$ for any $\varepsilon \in [0, \alpha]$. Note the following properties of $\delta(\varepsilon)$: (i) $\delta_1(0) = \frac{1}{2} - \gamma$ and (ii) $\delta_1(\alpha) = 0$.

Since preferences are continuous on $X = T_0^1$, $\delta_1(\varepsilon)$ is a continuous and monotonic function of ε. Define $g_1(\varepsilon) = \delta_1(\varepsilon) - \varepsilon$. Note that $g_1(0) = \frac{1}{2} - \gamma > 0$ and $g_1(\alpha) = -\alpha < 0$. Since $g_1(.)$ is a continuous function, we can apply the *intermediate value theorem* which implies that there exist ε_1^* such that $g_1(\varepsilon_1^*) = \delta_1(\varepsilon_1^*) - \varepsilon_1^* = 0$. This is illustrated in Figure 3(a).

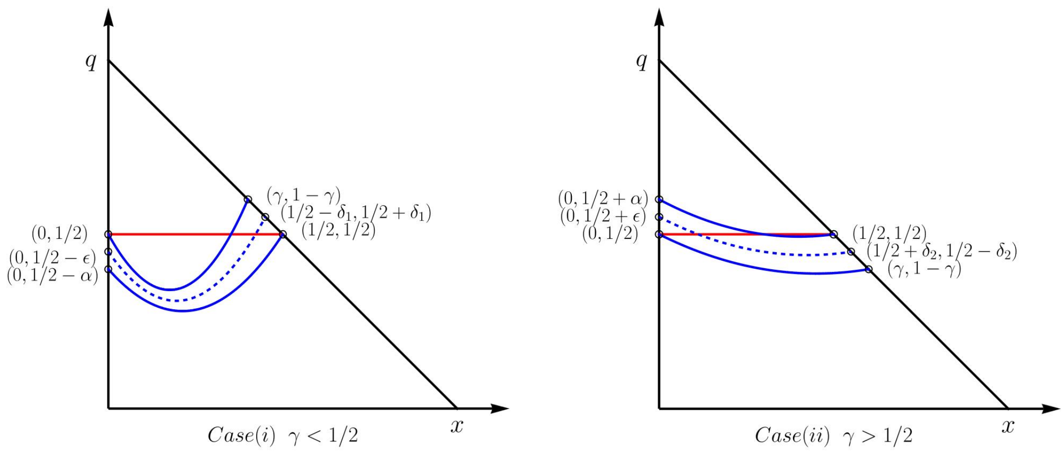

Figure 3: (a) and (b) : Proving existence of a balanced IC

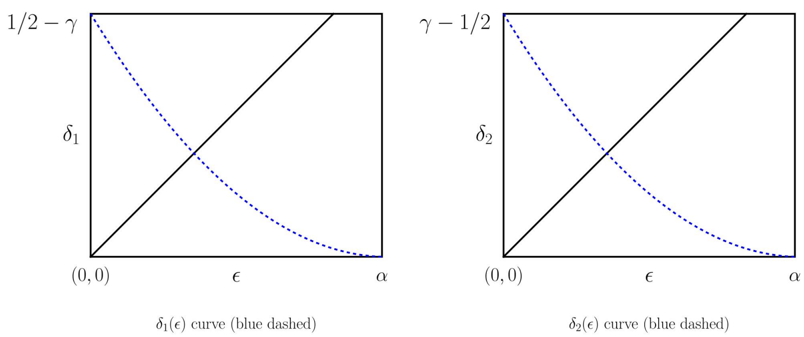

Figure 4: (a) and (b) illustrate $\delta_1(.)$ and $\delta_2(.)$ resp.

Therefore, $\delta_1(\varepsilon_1^*) = \varepsilon_1^*$ and this results in a contradiction since we assumed that such points do not exist. Therefore, there exist q_i^f with $IC_{i,q_i^f}(q_i^f) = 1 - q_i^f$ for each $i \in \{1, 2\}$.

Case (ii): $\frac{1}{2} < \gamma$, then for some $\alpha \in (0, \frac{1}{2})$, we will have $IC_{i,\frac{1}{2}+\alpha}(\frac{1}{2}) = \frac{1}{2}$. Now consider ICs between $IC_{i,\frac{1}{2}}(x)$ and $IC_{i,\frac{1}{2}+\alpha}(x)$ s.t. $IC_{i,\frac{1}{2}+\varepsilon}(\frac{1}{2} + \delta_2(\varepsilon)) = \frac{1}{2} - \delta_2(\varepsilon)$ for some function $\delta_2 : [0, \alpha] \to [0, \gamma - \frac{1}{2}]$ such that $\big(0, \frac{1}{2} + \varepsilon\big) \sim_i \big(\frac{1}{2} + \delta_2(\varepsilon), \frac{1}{2} - \delta_2(\varepsilon)\big)$ for any $\varepsilon \in [0, \alpha]$. Note the following properties of δ_2, (i) $\delta_2(0) = \gamma - \frac{1}{2}$ and (ii) $\delta_2(\alpha) = 0$.

Define $g_2(\varepsilon) = \delta_2(\varepsilon) - \varepsilon$. Note that $g_2(0) = \gamma - \frac{1}{2} > 0$ and $g_2(\alpha) = -\alpha < 0$. Since δ_2 is a continuous function, we can apply the *intermediate value theorem* which implies that there exist ε_2^* such that $g_2(\varepsilon_2^*) = \delta_2(\varepsilon_2^*) - \varepsilon_2^* = 0$. Therefore, $\delta_2(\varepsilon_2^*) = \varepsilon_2^*$ and this results in a contradiction since we assumed that such points do not exist. Therefore, there exist $q_i^f \in (0, 1)$ with $IC_{i,q_i^f}(q_i^f) = 1 - q_i^f$ for each $i \in \{1, 2\}$.

For ease of exposition $IC_{iq_i^f}(x) \equiv IC_i^f(x)$.

We now show that for each individual $i \in N$, q_i^f is unique. Suppose for contradiction that there are two balanced ICs: IC_i^f and $IC_i^{'f}$ and that w.l.o.g. $q_i^f < q_i^{'f}$ which implies $1 - q_i^{'f} < 1 - q_i^f$. However, this further implies that $IC_i^f(x)$ and $IC_i^{'f}(x)$ intersect. This is a contradiction to transitivity of the preferences. ■

Proposition 2 proves the existence of a unique balanced IC for any given preference $\succsim_i \in \mathcal{D}$. This will be used frequently to describe the algorithm to find envy-free and Pareto efficient algorithm. A term we will use is the balanced IC region which can be defined as follows. Suppose $q_1^f < q_2^f$ i.e. the balanced IC of agent 1 cuts the balanced IC of agent 2 from below. Note that no balanced IC can lie completely below or above another balanced IC. Any allocation $(0, \alpha)$ and $(\alpha, 1 - \alpha)$ is said to be in the balanced IC region if $\alpha \in (q_1^f, q_2^f)$.

Observation 1 *Suppose* $\mathbf{P} = (\succsim_1, \succsim_2)$ *such that* $\succsim_1 = \succsim_2 = \succsim$*, and* $\tau(\succsim_i) = \{0, 1\}$ *for each* $i \in \{1, 2\}$*, then any envy-free and Pareto efficient allocation* $f_1(\mathbf{P})$ *and* $f_2(\mathbf{P})$ *are both on the common balanced IC,* $IC_i^f(x)$*. In other words,* $f_1(\mathbf{P}) \sim_1 f_2(\mathbf{P})$ *and* $f_1(\mathbf{P}) \sim_2 f_2(\mathbf{P})$*.*

To see why the above claim is true, first note that both agents must receive allocations on the same IC: if the two allocations are not on the same IC, then the agent on the lower IC will envy the other. Suppose the allocation is not on the balanced IC for both the agents $IC_{1,q_1^f} = IC_{2,q_2^f}$. Then, any allocation cannot be on an IC above the balanced IC, since that would not be feasible. To see this, note that $(0, q_1^f)$ and $(q_1^f, 1 - q_1^f)$ are feasible. However, a higher $IC_{i,q}$ will be such that $q > q_1^f$, which means that $1 - \bar{q} > 1 - q_1^f$ where $\bar{q}$ is such that $(\bar{q}, 1 - \bar{q})$ is on $IC_{i,q}$ and intersects the line $x + q = 1$. But this implies that $q + 1 - \bar{q} > q_1^f + 1 - q_1^f = 1$. This implies that the allocation $f_1(\mathbf{P}), f_2(\mathbf{P}) \in \{(0,q),(\bar{q},1-\bar{q})\}$ for any $f_1(\mathbf{P}) \neq f_2(\mathbf{P})$ (which are on the same IC) is not feasible.

If the allocation of the agents lie on an IC below the balanced IC, then this is not Pareto efficient, since both would prefer receiving either of the two allocations $(0,q_1^f)$ or $(q_1^f, 1-q_1^f)$ on the balanced IC since it is on an IC closer to their top IC. They would both be strictly better-off since their top allocation is $(0,1) \in X$. Therefore, any no-envy and Pareto efficient allocation has to be on the balanced IC when the peak is at $(0,1)$ and both agents have the same preference. We will use a more general version of this observation in the proof of our main theorem.

Observation 2 *There exists a preference profile* $\mathbf{P} \in \mathcal{D}^2$ *for which there do not exist any allocation rule,* $f: \mathcal{D}^2 \to \mathcal{A}$*, that is Pareto efficient and envy-free.*

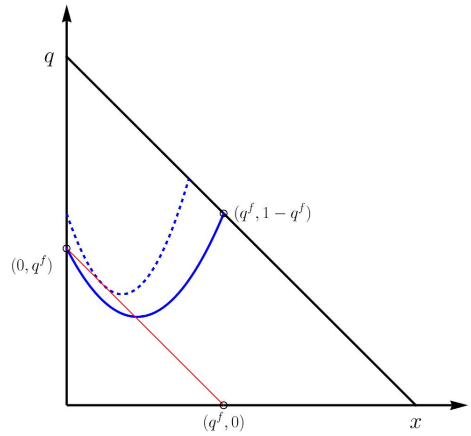

Figure 5: When preferences are "too steep"

We provide an example to prove the above observation. Consider a preference profile $\mathbf{P}$ where both agents have the same preference $\succsim_1 = \succsim_2 = \succsim$ and $\frac{d}{dx}(IC_{i,q}(x))|_{x=0} < -1$ for all $q \in (0,1)$ i.e. for every $IC_{i,q}$ there exists points x in the interior of the set $T_0^q = \{x \in T_0^1 \text{ s.t. } x+q<1\}$. In other words, some points on the $IC_{i,q}$ belong to the interior of the triangle T_0^q for every $q \in (0,1)$. By the above observation, a feasible envy-free and Pareto efficient allocation must be on the same IC. Here if any agent is allocated $(0,q^f)$, then she will take a subset of the interval, where the relevant IC through the allocation point is tangent. Therefore, the only possible allocation in this case, is where at the tangency point (x_1,q_1), the right cut-point of the IC through this point must be balanced, i.e., $(x_1,q_1) \sim_i (x_2, 1-q_2)$ for some q_2 such that $1-q_2 > 1-q^f$. However, this is not possible since $q_1 + 1 - q_2 > q^f + 1 - q^f = 1$ implies that the allocation $f_1(\mathbf{P}) = (x_1,q_1)$ and $f_2(\mathbf{P}) = (x_2,q_2)$ (or the allocation where the two agents get the permutation of this) is not feasible. In order to rule out such preference profiles, we impose another restriction preferences $\mathcal{D}$ to obtain a subset of the domain.

Assumption 4: For all $q \in (0,1)$, $\frac{\mathrm{d}}{\mathrm{d}x}(IC_{i,q}(0)) \geq -1$.

The reduced domain, $\mathcal{D}_1$**:** Let the set of preferences which satisfy Assumptions 1-3 be denoted as $\mathcal{D}_1$. Note that $\mathcal{D}_1 \subseteq \mathcal{D}$. We now highlight the importance of balanced ICs and the region between them by considering the case where preferences are **monotonic** i.e. the most preferred bundle for both agents is the full unit interval, i.e., $\tau(\succ_i) = \{(0,1)\}$ for $i \in \{1,2\}$.

Proposition 3 *Suppose* $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}_1^2$ *where* $\tau(\succsim_i) = \{(0,1)\}$ *for* $i \in \{1,2\}$. *Then the following allocation rule* $f : \mathcal{D}_1^2 \to \mathcal{A}$ *is Pareto efficient and envy-free:*
(i) If $q_1^f = q_2^f$*, then* $f_i(\mathbf{P}) = (0, q_1^f)$ *and* $f_j(\mathbf{P}) = (q_1^f, 1-q_1^f)$*, where* $i \neq j$*,* $i,j \in \{1,2\}$*.*
(ii) If $q_i^f < q_j^f$*, then* $f_i(\mathbf{P}) = (0,\alpha)$ *and* $f_j(\mathbf{P}) = (\alpha, 1-\alpha)$ *for all* $\alpha \in [q_i^f, q_j^f]$*.*

Proof. (i) If $q_1^f = q_2^f = q^f$ then either $f(\mathbf{P}) = ((0,q^f),(q^f,1-q^f))$ or $f(\mathbf{P}) = ((q^f,1-q^f),(0,q^f))$ is envy-free as both get allocations on same IC and Pareto efficient as whole resource is allocated.

(ii) We provide arguments for the case when $q_1^f < q_2^f$. Similar arguments can be used to prove the other case. Note that if $q_i^f \neq q_j^f$ then IC_{i,q_i^f} and IC_{j,q_j^f} can only intersect once i.e. $q_i^f < q_j^f \Rightarrow 1-q_i^f > 1-q_j^f$.

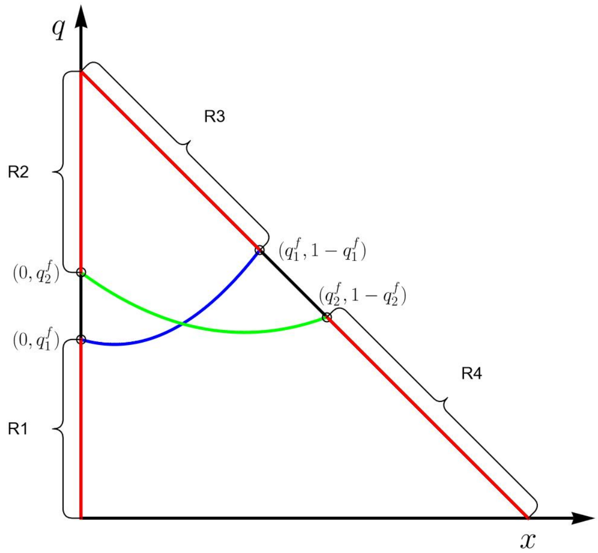

Figure 6: Illustrative balanced IC region

We define the following sets to argue that the provided allocations are envy-free and Pareto efficient: $R_1 = \{(0,\alpha) | \alpha \in [0, q_1^f)\}$, $R_2 = \{(0,\alpha)|\alpha \in (q_2^f, 1]\}$, $R_3 = \{(\alpha, 1-\alpha)|\alpha \in (q_1^f, 1]\}$ and $R_4 = \{(\alpha, 1-\alpha)|\alpha \in [0, q_2^f)\}$. We will first show that no Pareto efficient and envy-free allocation of the form $\{(0,\alpha),(\alpha,1-\alpha)\}$ can be in regions R_1, R_2, R_3 and R_4.

Consider any $(0,\alpha) \in R_1$, then the corresponding $(\alpha, 1-\alpha)$ will be in R_3. Here, both the agents will prefer $(\alpha, 1-\alpha)$ since their peak bundle is $(0, 1)$. Therefore, giving the latter to any agent will not be envy-free. Similarly, for any $(0, \alpha) \in R_2$, the corresponding $(\alpha, 1-\alpha)$ will be in R_4. Both the agents will prefer $(0, \alpha)$ so giving the latter will again lead to envy.

For any $(0, \alpha)$ s.t. $\alpha \in [q_1^f, q_2^f]$, corresponding $(\alpha, 1-\alpha)$ will be on $x + q = 1$ line between $(q_1^f, 1 - q_1^f)$ and $(q_2^f, 1 - q_2^f)$. We will have $(0, \alpha) \succ_1 (\alpha, 1-\alpha)$ and $(\alpha, 1-\alpha) \succ_2 (0, \alpha)$. Therefore, for any such allocation, neither of the two agents will envy the other's allocation. Moreover, this is Pareto efficient since there full interval is given away. ■

When both the agents have top ICs above region between the two balanced ICs and the domain of preferences is $\mathcal{D}_1$, any Pareto efficient allocation has to be a no-wastage allocation. In other words, any Pareto efficient allocation will assign allocations on the left and right axis respectively to the agents.

It is easy to verify the above observation. Once an allocation $(0, q)$ is given to an agent, the remaining feasible region is T_q^{1-q}. Since preferences are monotonic, for the other agent, the highest IC will intersect at $(q, 1-q)$. Similarly, if an agent gets an allocation $(q, 1-q)$ on the right axis, then the remaining feasible region is given by T_0^q. Given assumption 4 above, when more quantity is preferred, the highest IC of the other agent in T_0^q will intersect at $(0, q)$. We will say that two allocations $((x_1, q_1), (x_2, q_2))$ and $((x'_1, q'_1), (x'_2, q'_2))$ are *Pareto indifferent* to each other if both are Pareto efficient and $(x_1, q_1) \succsim_1 (x'_1, q'_1)$ and $(x'_2, q'_2) \succsim_2 (x_2, q_2)$.

We now describe the algorithm which will characterize the full set of Pareto efficient and envy-free allocation for any given preference profile $\mathbf{P} \in \mathcal{D}^2$.

Definition 1 (Balanced-curve allocation(BCA) rule) *An allocation rule $f : \mathcal{D}_1^2 \to \mathcal{A}$ is the BCA rule if for every $\mathbf{P} \in \mathcal{D}_1^2$ it produces an allocation according to the algorithm provided below.*

The balanced-cure allocation (BCA) rule as described below gives a set of possible allocations for each preference profile $\mathbf{P} \in \mathcal{D}_1^2$ based on the inequalities the individual preferences follow. Consider any preference profile $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}_1^2$. We will use the fact that there is a unique balanced IC for each agent denoted by IC_i^f for $i \in \{1, 2\}$.[10]

We say that $IC_{i,q} \leq IC_{i,q}$ when for all x, $IC_{i,q}(x) \leq IC_{i,q}(x)$. This is due to the fact that ICs do not intersect and equality holds when the ICs coincide.

Observation 3 *For any $i \in \{1, 2\}$,*

(i) (a) If $IC_i^\tau = IC_i^f$ then $q_i^\tau = q_i^f = \overline{q}_i^\tau$.

[10]Unless the preferences are identical in which case they have the same balanced IC.

(b) If $IC_i^\tau < IC_i^f$ then $q_i^\tau < q_i^f < \overline{q}_i^\tau$.

(c) If $IC_i^f < IC_i^\tau$ then $\overline{q}_i^\tau < q_i^f < q_i^\tau$.

(ii) If $1 - \overline{q}_j^\tau \leq 1 - q_i^\tau \Leftrightarrow q_i^\tau \leq \overline{q}_j^\tau$, then an allocation $\{(x_i, q_i)\}_{i=1}^2$ where $x_i < x_j$, we can give agent i, $(x_i, q_i) = (0, q_i^\tau)$, there is enough remaining for agent j to be allocated on it's top IC on the right of agent i.

We give a broad classification of the cases. There are three cases. Case 1 describes a profile when both agents can be allocated on their top ICs i.e. $q_1^\tau \leq \overline{q}_2^\tau$ or $q_2^\tau \leq \overline{q}_1^\tau$ or both. If only $q_1^\tau \leq \overline{q}_2^\tau$ but not $q_2^\tau \leq \overline{q}_1^\tau$, then $(0, q_1^\tau)$ and $(q_1^\tau, 1 - q_1^\tau)$ is a Pareto efficiency and envy-free allocation. However, there may be other Pareto efficient and envy-free allocations in this case. However, all such allocations will have the property that $x_1 < x_2$, i.e., agent 1 is given an allocation on the left of that of agent 2. These allocations will be denoted as LR allocations, where L denotes 'left' for agent 1, and R denotes 'right' for agent 2.

Similarly, if $q_2^\tau \leq \overline{q}_1^\tau$ but not $q_1^\tau \leq \overline{q}_2^\tau$ then $x_2 < x_1$ are the only Pareto efficient and envy-free allocations. These allocations will be denoted as RL allocations. But if both $q_1^\tau \leq \overline{q}_2^\tau$ and $q_2^\tau \leq \overline{q}_1^\tau$ then either configuration will result in allocations that are Pareto efficient and envy-free.

Cases 2 and 3 pertain to those profiles where one of the agents will not receive allocations on their top ICs. This implies that $\overline{q}_2^\tau < q_1^\tau$ and $\overline{q}_1^\tau < q_2^\tau$. Note that this implies that at least one agent has their top IC above the balanced IC i.e. for some $i \in 1, 2$, $IC_i^\tau(x) < IC_i^f(x)\ \forall x$. If both agents have their top IC below their balanced curve i.e, $q_1^\tau \leq q_1^f \leq \overline{q}_1^\tau$ and $q_2^\tau \leq q_2^f \leq \overline{q}_2^\tau$ but $\overline{q}_1^\tau < q_2^\tau$, will imply that $q_1^\tau \leq \overline{q}_2^\tau$. Similarly for $\overline{q}_2^\tau < q_1^\tau$ implies that in $q_2^\tau \leq \overline{q}_1^\tau$.

Case 2: The top IC of one agent is above its balanced IC and for the other the top IC is below its balanced IC. Case 3: The top ICs of both agents are above their respective balanced ICs. This results in allocations in the balanced band. Cases 3(i) and 3(ii) are when there exists allocations disjoint from allocations in the balanced band, 3(i) is case when top ICs intersect, while in 3(ii) they do not intersect. We now provide the details.

We note that the cases are mutually exclusive and exhaustive, since the inequalities are always of the form $q_i^\tau \leq q_i^f$ (or $q_i^f < q_i^\tau$), $q_i^\tau \leq \overline{q}_j^\tau$(or $\overline{q}_j^\tau < q_i^\tau$), $\overline{q}_i^\tau \leq \overline{q}_j^\tau$(or $\overline{q}_j^\tau < \overline{q}_i^\tau$) and $q_i^\tau \leq q_j^\tau$ (or $q_j^\tau < q_i^\tau$).

Case 1: $q_1^\tau \leq \overline{q}_2^\tau$ or $q_2^\tau \leq \overline{q}_1^\tau$ or both. Here, both the agents get an allocation on their top IC. There are three sub-cases. Case 1(i): $q_1^\tau \leq \overline{q}_2^\tau$ and $q_2^\tau \leq \overline{q}_1^\tau$. In this case, both LR and RL type allocations are possible. LR: If $q_1^\tau = \overline{q}_2^\tau$ then $f_1(P) = (0, q_1^\tau)$, and $f_2(P) = (\overline{q}_2^\tau, 1 - \overline{q}_2^\tau)$, otherwise, define $\tilde{x}$ such that $\tilde{x} + IC_1^\tau(\tilde{x}) = \overline{q}_2^\tau$.

$f_1(\mathbf{P}) = (x_1, IC_1^\tau(x_1))$ where $x_1 \in [0, \tilde{x}]$ and $f_2(\mathbf{P}) = (x_2, IC_2^\tau(x_2))$ where $x_2 \in [x_1 + IC_1^\tau(x_1), \overline{q}_2^\tau]$.

RL: If $q_2^\tau = \overline{q}_1^\tau$ then $f_2(P) = (0, q_2^\tau)$ and $f_1(P) = (\overline{q}_1^\tau, 1 - \overline{q}_1^\tau)$, otherwise, define $\tilde{x}$ such that $\tilde{x} + IC_2^\tau(\tilde{x}) = \overline{q}_1^\tau$. Let $f_2(\mathbf{P}) = (x_2, IC_2^\tau(x_2))$ where $x_2 \in [0, \tilde{x}]$ and $f_1(\mathbf{P}) = (x_1, IC_1^\tau(x_1))$ where $x_1 \in [x_2 + IC_2^\tau(x_2), \overline{q}_1^\tau]$.

Case 1(ii): Suppose $q_1^\tau \leq \overline{q}_2^\tau$ and $\overline{q}_1^\tau < q_2^\tau$. Here only LR allocations are possible and these allocations are the same as in LR for case 1(i) above. RL allocations are not Pareto efficient since both agents cannot be provided on the top in an RL allocations. Moreover, any such envy-free RL allocation can be Pareto improved by giving the agents an LR allocation as described above.

Case 1(iii): Suppose $\overline{q}_2^\tau < q_1^\tau$ and $q_2^\tau \leq \overline{q}_1^\tau$. Here, there are only RL allocations, which are the same as in RL allocations for case 1(i) above. Any envy-free LR allocations will not be Pareto efficient since neither agent will get an allocation on her top IC.

Cases 2 and 3 consider the case where $\overline{q}_2^\tau < q_1^\tau$ and $\overline{q}_1^\tau < q_2^\tau$. In this case, there are no envy-free and Pareto efficient allocations where both the agents get an allocation on the top IC simultaneously. Pareto efficiency that a no-wastage allocations will be envy-free when both the agents cannot be allocated simultaneously on their top IC, i.e., when $1 - q_1^\tau < 1 - \overline{q}_2^\tau$ and $1 - q_2^\tau < 1 - \overline{q}_1^\tau$. If agent i is allocated on the left of agent $j \neq i$ then all allocations will be of the form $f_i(\mathbf{P}) = (0, a)$ and $f_j(\mathbf{P}) = (a, 1 - a)$ where $a \in [\overline{q}_j^\tau, q_i^\tau]$. If $a < \overline{q}_j^\tau$, then some resource can be transferred from agent j to i as agent j has more than required to get allocation on her top IC. Similarly if $q_i^\tau < a$, some resource can be transferred from agent i to j as agent i has more than required to get allocation on top IC.

Case 2: Suppose $IC_i^\tau \leq IC_i^f$ for some $i \in \{1, 2\}$ and $IC_j^f < IC_j^\tau$, for $j \neq i$, and $q_i^\tau \leq q_i^f \leq \overline{q}_i^\tau$ and $\overline{q}_j^\tau < q_j^f < q_j^\tau$. The inequalities imply that $\overline{q}_j^\tau < q_i^\tau \leq \overline{q}_i^\tau < q_j^\tau$. The following allocations are allocated under the BCA:

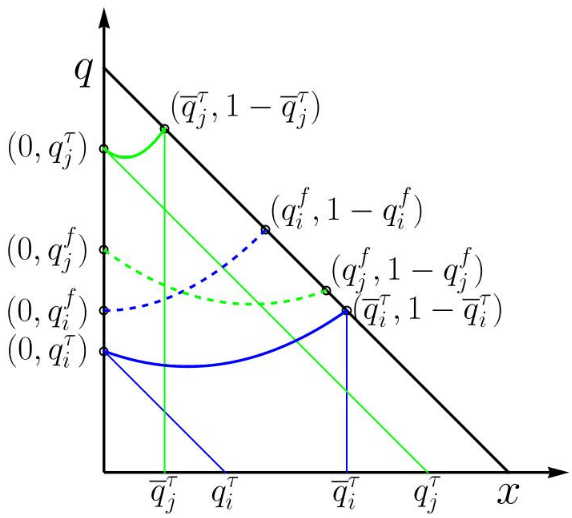

Figure 7: $\overline{q}_j^\tau < q_i^\tau \leq \overline{q}_i^\tau < q_j^\tau$

(a) Left-allocation to agent i and right-allocation to agent j: $f_i(\mathbf{P}) = (0, a)$ and $f_j(\mathbf{P}) = (a, 1-a)$ where $a \in [\alpha, \beta]$ such that $\beta = \min\{q_j^f, q_i^\tau\}$ and $\alpha = \min\{b \in [\overline{q}_j^\tau, \beta) : (0, a) \succsim_i (a, 1-a)\}$. (b) Left-allocation to agent j and right-allocation to agent i: $f_i(\mathbf{P}) = (a, 1-a)$ and $f_j(\mathbf{P}) = (0, a)$ where $a \in [\alpha, \beta]$ such that $\alpha = \max\{q_j^f, \overline{q}_i^\tau\}$ and $\beta = \max\{b \in (\alpha, q_j^\tau] : (a, 1-a) \succsim_i (0, a)\}$.

Case 3: The top IC of both agents is above the balanced IC of the corresponding agent, i.e., $IC_i^f < IC_i^\tau$ for both agents $i \in \{1, 2\}$, and $\overline{q}_1^\tau < q_1^f < q_1^\tau$ and $\overline{q}_2^\tau < q_2^f < q_2^\tau$.

In this case there are always some allocations in the balanced band.

Case 3(i): If $q_i^f = q_j^f = q^f$, then allocated $f_i(\mathbf{P}) = (0, q^f)$ and $f_j(\mathbf{P}) = (q^f, 1-q^f)$ **or** $f_j(\mathbf{P}) = (0, q^f)$ and $f_i(\mathbf{P}) = (q^f, 1-q^f)$. If $q_1^f < q_2^f$, then in balanced region allocation of agent 1 has to be on the left and similarly if $q_2^f < q_1^f$ then agent 2's allocation has to be on the left. We specify these allocations below.

a) If $q_1^f = q_2^f = q^f$, then $f_1(\mathbf{P}) = (0, q^f)$ and $f_2(\mathbf{P}) = (q^f, 1-q^f)$ or $f_2(\mathbf{P}) = (0, q^f)$ and $f_1(\mathbf{P}) = (q^f, 1-q^f)$.

b) If $q_i^f < q_j^f$ then allocate, $f_i(\mathbf{P}) = (0, a)$ and $f_j(\mathbf{P}) = (a, 1-a)$ where $a \in [\max\{\overline{q}_j^\tau, q_i^f\}, \min\{q_i^\tau, q_j^f\}]$.

Apart from the allocations in the balanced band there can be allocations outside it, if the following hold.

Case 3(ii): both agents top ICs **cross**, $i \neq j$, $\overline{q}_i^\tau \leq \overline{q}_j^\tau$ ($\equiv 1 - \overline{q}_j^\tau \leq 1 - \overline{q}_i^\tau$) and $q_i^\tau \leq q_j^\tau$ and ($\max\{q_i^f, q_j^f\} < q_i^\tau$ **or** $\overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$). Above inequalities imply $\overline{q}_i^\tau \leq \overline{q}_j^\tau < q_i^\tau \leq q_j^\tau$. Apart from above allocations we have the following additional allocations,

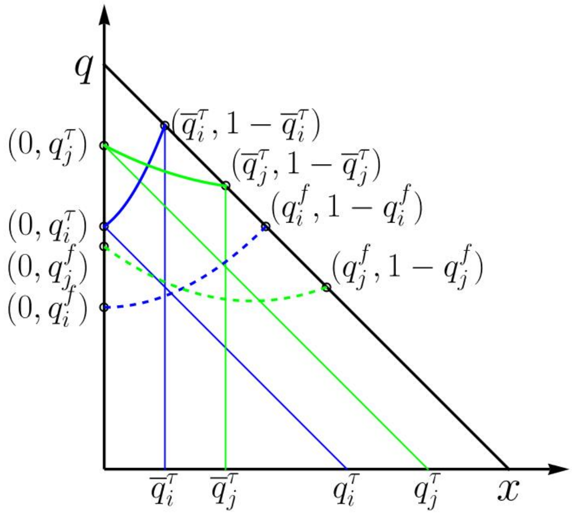

Figure 8: $\overline{q}_i^\tau \leq \overline{q}_j^\tau < q_i^\tau \leq q_j^\tau$

(a) If $\max\{q_i^f, q_j^f\} < q_i^\tau$, $f_i(\mathbf{P}) = (a, 1-a)$ and $f_j(\mathbf{P}) = (0, a)$ where $a \in (q_i^\tau, q_j^\tau]$ s.t. $(a, 1-a) \succsim_i (0, a)$.
(b) If $\overline{q}_j^\tau < \min\{q_i^f, q_j^f\})$, $a \in [\overline{q}_i^\tau, \overline{q}_j^\tau)$ s.t. $(0, a) \succsim_j (a, 1-a)$.

Case 3(iii): Both agents top ICs do not cross, $i \neq j$, $\overline{q}_i^\tau \leq \overline{q}_j^\tau$ ($\equiv 1 - \overline{q}_j^\tau \leq 1 - \overline{q}_i^\tau$) and $q_j^\tau < q_i^\tau$ and ($\max\{q_i^f, q_j^f\} < q_j^\tau$ or $\overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$). Above inequalities imply $\overline{q}_i^\tau \leq \overline{q}_j^\tau < q_j^\tau < q_i^\tau$.

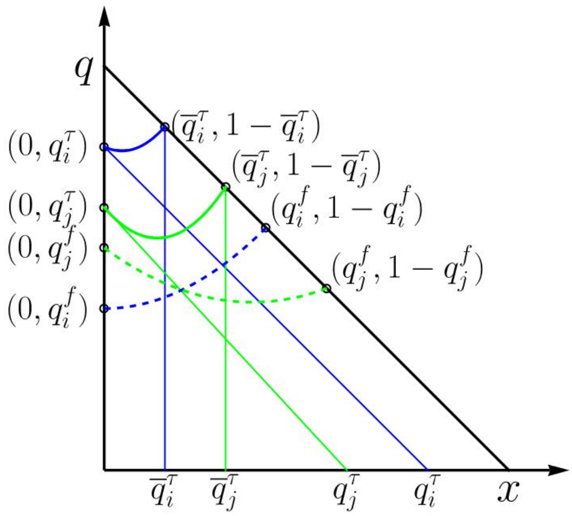

Figure 9: $\overline{q}_i^\tau \leq \overline{q}_j^\tau < q_j^\tau < q_i^\tau$

(a) If $\max\{q_i^f, q_j^f\} < q_j^\tau$, BCA provides the following allocations: $f_i(\mathbf{P}) = (0, a)$ and $f_j(\mathbf{P}) = (a, 1-a)$ where $a \in (q_j^\tau, q_i^\tau]$ s.t. $(a, 1-a) \succsim_j (0, a)$.
(b) If $\overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$, then the BCA gives the following allocations: $f_i(\mathbf{P}) = (a, 1-a)$ and $f_j(\mathbf{P}) = (0, a)$ where $a \in [\overline{q}_i^\tau, \overline{q}_j^\tau)$ such that $(0, a) \succsim_j (a, 1-a)$.

Remark 1 *Note that the BCA rule is not single-valued and for a given profile, the BCA provides a (non-empty) multi-valued set of allocations. The following theorem states that BCA allocations described above are the only Pareto efficient and envy-free allocations.*

Theorem 1 *An allocation rule $f : \mathcal{D}^2 \to \mathcal{A}$ is Pareto efficient and envy-free if and only if it is the BCA.*

We have shown in Proposition 3 that if preferences are monotonic i.e. if both agents prefer the whole interval to any other allocation, then Pareto efficient and envy-free allocations must be in the region between the respective balanced ICs of the two agents where the agent whose balanced IC cuts the other's balanced IC from below gets allocations on the left axis and the other gets the remaining on the right axis. We write this formally below.

Due to assumption 4, an agent is either provided an allocation on the right, i.e., allocation of the type $(a, 1-a)$, or on the left, i.e., an allocation of the type $(0, a)$ when both prefer to have the maximum quantity (Observation 3). In all the other cases, if an agent (on the right axis) needs less than the quantity provided in the balanced IC region, she is given up to her top IC and quantity is decreased from her top IC to the infimum (or supremum) of the location of the allocations which that agent prefers to the allocations given to the other agent. This prevents envy while maintaining Pareto efficiency. The proof of Theorem 1 for separate cases using the BCA described earlier.

Proof. We use a Lemma to prove our result. The Lemma states that for any top IC which connects $(0, q_i^\tau)$, there exists a q' such that the bundle on the right axis, $(0, q')$ is indifferent to $(0, q)$.

Lemma 1 *If for any $i \in N$, $q_i^\tau \in (0, 1)$, for all $q \in (q_i^\tau, 1]$ there exists $q' \in (0, q_i^\tau)$ such that $(0, q) \sim_i (0, q')$*

Proof. By assumption 2 we know that $(0, 1) \succ_i (0, 0)$. By single-peakedness in quantity (assumption 3) and assumption 2: for any $q \in (q_i^\tau, 1)$, $(0, q_i^\tau) \succ_i (0, q) \succ_i (0, 1) \succ_i (0, 0)$. By continuity of preferences (assumption 1) there will be $q_1, q' \in (0, q_i^\tau)$ and $0 < q_1 < q' < q_i^\tau < q < 1$ such that $(0, q_1) \sim_i (0, 1)$ and $(0, q') \sim_i (0, q)$. ■

Cases. We prove in separate cases, that each of the allocations prescribed by the BCA in the definition of the rule are the only Pareto efficient and envy-free allocations. For sake of simplicity, we do not reiterate the conditions under different cases and the allocations, we simply provide the proofs of why such allocations are the only Pareto indifferent and envy-free allocations.

Case 1. Case 1(i): LR allocation: These allocations are Pareto efficient and envy-free as both agents get allocation on their top ICs. Giving $f_1(\mathbf{P}) = (0, q_1^\tau)$, leaves more than enough for agent 2 on its top IC, since $1 - \overline{q}_2^\tau \leq 1 - q_1^\tau$.

RL allocation: These allocations are Pareto efficient and envy-free as both agents get allocation on their top ICs. Giving $f_2(\mathbf{P}) = (0, q_2^\tau)$, leaves more than enough for agent 1 on its top IC, since $1 - \overline{q}_1^\tau \leq 1 - q_2^\tau$.

Case 1(ii): Here, only LR allocations are Pareto efficient and envy-free. Since $1 - q_2^\tau < 1 - \overline{q}_1^\tau$, in a RL allocation, both agents can not get an allocation on their top IC and hence it is not Pareto indifferent to the LR allocation.

Case 1(iii): Here, only RL allocations are Pareto efficient and envy-free. Because in LR $1 - q_1^\tau < 1 - \overline{q}_2^\tau$ both agents can not get allocation on their top IC and hence it is not PI to RL. No other set of allocations can be Pareto efficient and envy-free since in all the allocations above, both the agents get an allocation on their respective top IC.

Case 2. Left-allocation to Agent i, Right-allocation to Agent j: Since agent i gets $(0, a)$, when $a < q_i^\tau$, the quantity given to i must be at the minimum level where she does not envy agent j as allocations are on opposite side of IC_i^τ. This implies that $a \in [\alpha, \beta]$ such that $\beta = \min\{q_j^f, q_i^\tau\}$ and $\alpha = \min\{b \in [\overline{q}_j^\tau, \beta) : (0, a) \succ_i (a, 1 - a)\}$. If $a < \alpha$, then agent i will envy agent j, and if $a > \beta$, then either (i) $q_i^\tau > q_j^f$: in this case, agent j will envy agent 1 since the latter's allocation will be closer to the top IC of agent j than her own allocation, or (ii) $q_i^\tau \leq q_j^f$: in this case, agent i gets more than her top quantity and this cannot be Pareto efficient.

Left-allocation to agent j, and right-allocation to agent i: Here agent i gets an allocation $(a, 1-a)$, while agent j gets $(0, a)$, where $a \in [\alpha, \beta]$ such that $\alpha = \max\{q_j^f, \overline{q}_i^\tau\}$ and $\beta = \max\{b \in (\alpha, q_j^\tau] : (a, 1 - a) \succ_i (0, a)\}$. Here, if $a < \alpha$, then if (i) $q_j^f > \overline{q}_i^\tau$, then agent j will envy i if $a < q_j^f$, and if (ii) $q_j^f \leq \overline{q}_i^\tau$, then $a < \overline{q}_i^\tau$ will not be Pareto optimal since we would be giving a quantity of $1 - a$ to agent i, which more than she needs to be on her top IC. Similarly, if $a > \beta$, then agent i will envy agent j since $(0, a) \succ_i (a, 1 - a)$.

Case 3: Case 3(i): If $q_i^f = q_j^f = q^f$, then both the agents are allocated on the "common" end points of balanced ICs. Therefore, there is no envy, and since the whole resource is allocated, it is Pareto efficient. Any other allocation will cause envy, by similar arguments as in Proposition 3.

Allocations in the balanced region when $q_i^f \neq q_j^f$: If $q_i^f < q_j^f$, in the LR allocation, $a < q_i^f$ will imply that agent i will envy agent j and $a < \overline{q}_j^\tau$ will not be Pareto efficient as agent j will have more than required for allocation on top IC. Hence, $\max\{\overline{q}_j^\tau, q_i^f\} \leq a$. If $q_j^f < a$, then agent j will envy agent i and if $q_i^\tau < a$ the allocation will not be Pareto efficient as agent i will have more than required for allocation on top IC. Hence, $a \leq \min\{q_i^\tau, q_j^f\}$.

If $q_j^f < q_i^f$: in the RL allocation, if $a < q_j^f$, then agent j will envy i and if $a < \overline{q}_i^\tau$, then the allocation is not Pareto efficient as agent i will have more than required for allocation on top IC. Hence, $\max\{\overline{q}_i^\tau, q_j^f\} \leq a$. If $q_i^f < a$, then agent i will envy agent j and if $q_j^\tau < a$ the allocation will not be Pareto efficient as agent i will have more than required for allocation on top IC. Hence, $a \leq \min\{q_j^\tau, q_i^f\}$.

Case 3(ii): If $a = \overline{q}_j^\tau$, then agent j will envy agent i as agent i has allocation on IC_j^τ but for $a < \overline{q}_j^\tau$, the allocation of i and j are on opposite sides of IC_j^τ. Therefore, $a > q_j^\tau$ given that agent j does not envy agent i. Agent i will not envy j as it's allocation is above IC_i^f while j's allocation is below IC_i^f. If $a = q_i^\tau$, then agent i will envy agent j as agent j has allocation on IC_i^τ. If $q_i^\tau < a$, the allocation of i and j are on opposite sides of IC_i^τ. Therefore, only allocations where agent i does not envy agent j are allowed. Agent j will not envy as it's allocation is above IC_j^f while i's allocation is below IC_j^f.

Case 3(iii): Left-allocation to agent i and right-allocation to agent j: If $q_j^\tau = a$, then agent j will envy agent i as agent i has an allocation on IC_j^τ but for $q_j^\tau < a$, the allocation of i and j are on opposite sides of IC_j^τ. Therefore, $a > q_j^\tau$. Agent i will not envy as it's allocation is above IC_i^f while j's allocation is below IC_i^f. By similar arguments as above, we must have $a \in (q_j^\tau, q_i^\tau]$. Left-allocation to agent j and right-allocation to agent i. When $f_i(\mathbf{P}) = (a, 1-a)$ and $f_j(\mathbf{P}) = (0, a)$: If $\overline{q}_j^\tau = a$, then agent j will envy agent i as agent i has allocation on IC_j^τ. If $a < \overline{q}_j^\tau$, the allocation of i and j are on opposite sides of IC_j^τ, hence only allocations where agent j does not envy agent i are allowed. Agent i will not envy as it's allocation is above IC_i^f while j's allocation is below IC_i^f. ■

Theorem 1 provides a characterization of envy-free and Pareto efficient allocations. However, the class of rules under BCA are not single-valued and We now show that adding strategy-proofness to the above axioms gives us an impossibility.

Theorem 2 *There is no allocation rule $f : \mathcal{D}_1^2 \to \mathcal{A}$ that is strategy-proof, envy-free and Pareto efficient.*

Proof. Suppose $f : \mathcal{D}_1^2 \to \mathcal{A}$ is strategy-proof, envy-free and Pareto efficient. Consider the following three types of preferences where the peak allocation is $\{(0,1)\}$ and all the ICs are *linear*. Type 1: $\succsim_i$, characterized by ICs with slope 0 i.e. $\frac{d}{dx}(IC_{i,q}(x)) = 0$ for all $q \in [0,1)$ and $x \in (0,1)$, Type 2: $\succsim_i'$, characterized by ICs with positive slope (except for $IC_{i,0}'$), i.e., $\frac{d}{dx}(IC_{i,q}(x)) > 0$ for all $q \in (0,1)$ and $x \in (0,1)$ and Type 3: $\succsim_i''$, with all ICs of negative slope (except for $IC_{i,0}''$) i.e., $\frac{d}{dx}(IC_{i,q}(x)) < 0$ for all $q \in (0,1)$ and $x \in (0,1)$. The actual value of the slope does not matter as long as it is positive (or negative) in the corresponding regions. Let q^f, q'^f and q''^f be the respective quantities of the balanced ICs at $x = 0$.

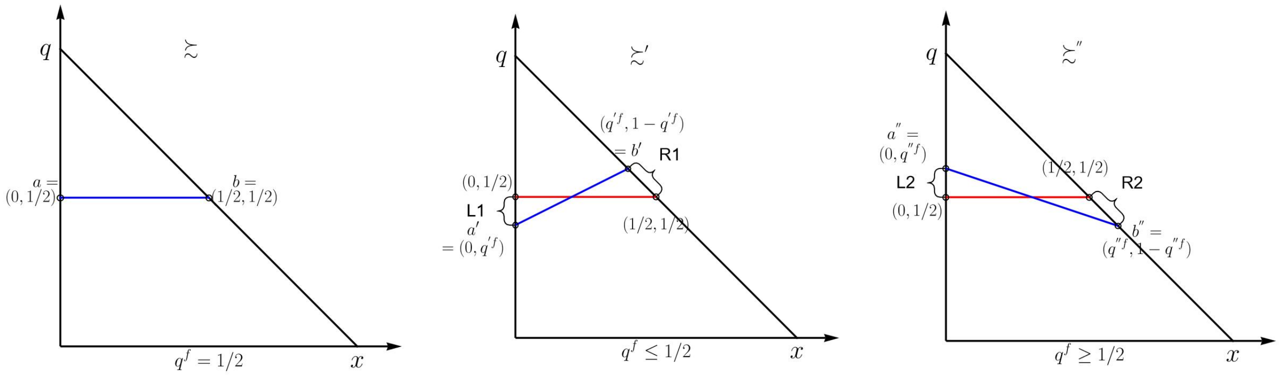

Figure 10: Construction of preferences for Theorem 2

We define the sets, $L_1 = \{(0, q) | q'^f \leq q \leq \frac{1}{2}\}$, $L_2 = \{(0, q) | \frac{1}{2} \leq q \leq q''^f\}$, $R_1 = \{(q, 1 - q) | \frac{1}{2} \leq q \leq q'^f\}$ and $R_2 = \{(q, 1 - q) | q''^f \leq q \leq \frac{1}{2}\}$. To prove our claim we construct profiles that consist of preferences from the set $\{\succsim_i, \succsim'_i, \succsim''_i\}$.

Let $\mathbf{P}^1 = \{\succsim_1, \succsim_2\}$. Since f is envy-free and Pareto efficient, $f_1(\mathbf{P}^1) = a = (0, \frac{1}{2})$ and $f_2(\mathbf{P}^1) = b = (\frac{1}{2}, \frac{1}{2})$. Let $\mathbf{P}^2 = \{\succsim'_1, \succsim_2\}$. According to Lemma 3, agent 1 has to be allocated in L_1 and agent 2 in R_1, to prevent envy. To prevent agent 1 from deviating from $\mathbf{P}^2$ to $\mathbf{P}^1$ via $\succsim_1$, we have $f_1(\mathbf{P}^2) = a$ and $f_2(\mathbf{P}^2) = b$. Note that $q_1'^f = q_2'^f = q'^f$ and $q_1''^f = q_2''^f = q''^f$.

Suppose $\mathbf{P}^3 = \{\succsim'_1, \succsim'_2\}$. Since both have identical preferences, envy-freeness requires that allocations be on either ends of the corresponding balanced IC i.e. $(0, q'^f)$ and $(q'^f, 1 - q'^f)$. To prevent agent 2 from deviating from profile $\mathbf{P}^2$ to profile $\mathbf{P}^3$ via $\succsim'_2$, by strategy-proofness $f_2(\mathbf{P}^3) = a'$. Note, $a' \succ'_2 b$ and $b' \succ_2 b \succ_2 a'$. By Pareto efficiency, $f_1(\mathbf{P}^3) = b'$.

Let $\mathbf{P}^4 = \{\succsim_1, \succsim'_2\}$. Here, according to Lemma 3, agent 2 has to be allocated in L_1 and agent 1 in R_1. To prevent agent 1 from deviating from $\mathbf{P}^4$ to $\mathbf{P}^3$ via $\succsim'_1$, $f_1(\mathbf{P}^4) = b'$. By Pareto efficiency, $f_2(\mathbf{P}^4) = a'$. Suppose profile $\mathbf{P}^5 = \{\succsim''_1, \succsim'_2\}$. By Lemma 3, agent 2 has to be allocated in $L_1 \cup L_2$ and agent 1 in $R_1 \cup R_2$. To prevent agent 1 from deviating from $\mathbf{P}^5$ to $\mathbf{P}^4$ via $\succsim'_1$, by strategy-proofness and Pareto efficiency, $f_1(\mathbf{P}^5) = b'$ and $f_2(\mathbf{P}^5) = a'$.

Finally, let $\mathbf{P}^6 = \{\succsim''_1, \succsim_2\}$. According to Proposition 3 and envy-freeness, agent 2 must be allocated in L_2 and agent 1 in R_2. But this will result in agent 2 deviating at $\mathbf{P}^5$ to $\mathbf{P}^6$ via $\succsim_2$ where her allocation will be strictly better since $f(\mathbf{P}^6) = (0, \alpha) \succ'_2 a'$ for any $\alpha \in [\frac{1}{2}, q''_f]$. This is a contradiction to the fact that f is strategy-proof. ■

5 More than 2 agents

We first provide an algorithm to obtain an envy-free and Pareto efficient allocations for 3 agents when all the agents have preference for greater quantity i.e. $\tau(\succsim_i) = \{(0,1)\}$ for each $i \in N \equiv \{1,2,3\}$ and for *linear preferences.* A preference $\succsim_i \in \mathcal{D}_1$ is *linear* if it is represented by ICs which satisfy $\frac{d}{dx}(IC_{i,q}(x)) = a_q$ for some $a_q \in \mathbb{R}$ for any given $q \in (0,1)$ for all $x \in (0, \overline{q}_i^{\tau})$. Let this domain of preference be denoted by $\mathcal{D}_2 \subset \mathcal{D}_1 \subset \mathcal{D}$. An allocation rule for $k \geq 3$ agents is a mapping $f : \mathcal{D}_2^k \to \mathcal{A}$.

A *k-balanced IC* for an agent $i \in N = \{1,2,3\}$ is an IC, IC_i^f such that there exist three allocations $(0, q_{i1})$, (x_{i2}, q_{i2}) and (x_{i3}, q_{i3}) where $0 = x_{i1} < x_{i2} < x_{i3}$, $x_{i2} = x_{i1} + q_{i1}, x_{i3} = x_{i2} + q_{i2}$ and $q_{i1} + q_{i2} + q_{i3} = 1$. The existence of a k-balanced IC can be proved using similar arguments as the ones used in Proposition 2. We provide only a sketch of the proof. For any preference $\succsim_i \in \mathcal{D}_1$ and any given IC, $IC_{i,q}$ which represents it, we can construct a left-over or wastage function $\delta(x(q))$ as done for 2 agents earlier where x is the length of the portion left-over (surplus or deficit) after the first $k-1$ allocations $a_1, ..., a_{k-1}$ all lie on the $IC_{i,q}$. More specifically, if $q_1, .., q_{k-1}$ are the quantities in the allocations $a_i = (x_i, q_i)$ for $i \in \{1, ..., k-1\}$ then the wastage is given by $x(q) = 1 - \sum_{i=1}^{k-1} q_i$.[11]

Clearly, an implication of our Assumptions 1-4 in Section 2 is that as q approaches zero, we can find $q \in (0,1)$ and an $IC_{i,q}$ such that $\delta(x(q)) > 0$ (since the slope of ICs approach zero). Similarly, we can always find another $q' \in (0,1)$ such that $\delta(x(q')) < 0$. By continuity of preferences, $\delta(x(q))$ is a continuous function of $x(.)$ and $x(q)$ is a continuous function of q. Therefore, by the intermediate value theorem, there exists a $\hat{q} \in (0,1)$ such that $\delta(x(\hat{q})) = 0$.

Balanced curve algorithm for 3 agents (BCA-3) Suppose the 3 pieces balanced ICs are $IC_{1,q_1^f}, IC_{2,q_2^f}$ and IC_{3,q_3^f} have the slopes $a_{q_1^f} > a_{q_2^f} > a_{q_3^f}$ respectively. Let $a_1 = (x_{21}, q_{21})$, $a_2 = (x_{22}, q_{22})$ and $a_3 = (x_{23}, q_{23})$ be the three allocations that lie on agent 2's balanced IC, IC_{2,q_2^f}. Then the following allocation $f_1(\mathbf{P}) = a_1, f_2(\mathbf{P}) = a_2$ and $f_3(\mathbf{P}) = a_3$ is envy-free and Pareto efficient for any profile $\mathbf{P} \in \mathcal{D}_2^3$ i.e. for linear and monotone preferences.

[11] For very high ICs it may be the case that strictly fewer than $k-1$ allocations can be allocated on the given IC. In such cases, we can define $\delta(x(q)) = 1 - \sum_{i=1}^{k(q)}$ where $k(q) \in \{1, ..., k-1\}$ is the maximum number of allocations that can be given on $IC_{i,q}$ which are fewer or equal to $k-1$.

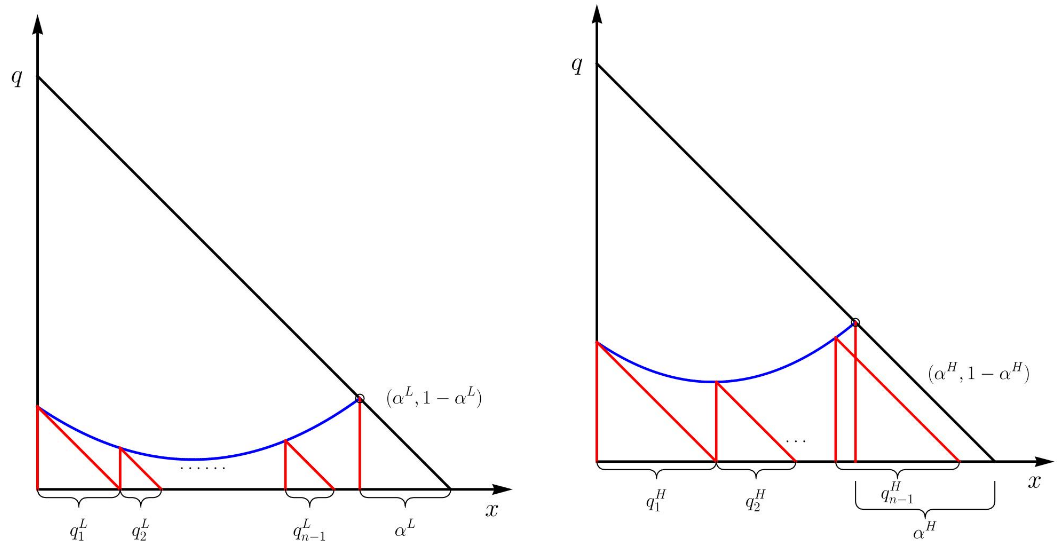

Figure 11: Balanced IC for k pieces

A similar algorithm can be constructed for 4 agents. We provide a property that must be satisfied for any envy-free and Pareto efficient allocation when preferences are linear and monotonic.

Proposition 4 *For any profile* $\mathbf{P} \in \mathcal{D}_2^n$ *with* $n \geq 2$*, suppose the allocation* $\{a_i\}_{i=1}^n = \{x_i, q_i\}_{i=1}^n$ *is envy-free and Pareto efficient. For simplicity assume that* $x_1 < x_2 < ... < x_n$ *and let the slope of an agent* i*'s IC through her own allocation* a_i *be* a_{q_i}*. Then it must be the case that* $a_{q_1} \geq a_{q_2} \geq .... \geq a_{q_n}$*.*

Proof. Suppose for contradiction that agent 1's IC's slope a_{q_1} is not the largest value through her own allocation $(0, q_1)$. By envy-freeness, all the other allocations must lie below agent 1's IC through $(0, q_1)$. But if agent $k \neq 1$ has a greater slope than her allocation (x_k, q_k) which lies below the IC of agent 1 through her own allocation, then agent k will prefer to have the allocation $(0, q_1)$ since her IC through the latter's allocation will be higher than the IC through her own allocation. This is a contradiction to the fact the allocation is envy-free. Similar arguments show that agent 2's IC through her own allocation must have a greater slope than all the succeeding agents' ICs through their respective allocations. ∎

The above proposition provides a necessary condition for envy-free and Pareto efficient when preferences are monotonic in quantity and linear for more than 2 agents. It states that the slopes of the ICs of agents when their allocations are placed from left to right must also be decreasing in the same direction. A violation of this would result in envy, since one of the ICs of an agent would pass through the allocation of another agent and lie completely above the IC of that agent through her own allocation. The above propositions provide insights into the more general problem of fair allocation with more than 2 agents.

6 Conclusion

We consider an allocation model where agents have a preference for location and quantity and the preference for quantity is single-peaked. We characterize the set of envy-free and Pareto efficient allocations. We show that there do not exist any strategy-proof, envy-free and Pareto efficient allocation rules. We provide observations for more than 2 agents which can be used to extend the BCA. The existence of the balanced IC for k portions may also prove beneficial for obtaining a complete characterization of envy-free rules for more than 2 agents.

For future research, other preference restrictions on the unit interval can be explored. However, the existence of a balanced IC might be crucial in such cases as well since the agents must be allocated on the same IC when they have identical preferences. Other extensions include a model where agents are allowed to trade between themselves, given a set of endowments.

References

Aumann, Y. and Y. Dombb (2015): "The efficiency of fair division with connected pieces," *ACM Transactions on Economics and Computation (TEAC)*, 3, 1–16.

Aziz, H. and S. Mackenzie (2016): "A discrete and bounded envy-free cake cutting protocol for any number of agents," in *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, IEEE, 416–427.

Bogomolnaia, A. and H. Moulin (2023): "Guarantees in fair division: general or monotone preferences," *Mathematics of Operations Research*, 48, 160–176.

Brams, S. J., M. A. Jones, and C. Klamler (2013): "N-person cake-cutting: There may be no perfect division," *The American Mathematical Monthly*, 120, 35–47.

Brams, S. J., A. D. Taylor, and W. S. Zwicker (1995): "Old and new moving-knife schemes," *The Mathematical Intelligencer*, 17, 30–35.

Caragiannis, I., C. Kaklamanis, P. Kanellopoulos, and M. Kyropoulou (2012): "The efficiency of fair division," *Theory of Computing Systems*, 50, 589–610.

Chen, Y., J. K. Lai, D. C. Parkes, and A. D. Procaccia (2013): "Truth, justice, and cake cutting," *Games and Economic Behavior*, 77, 284–297.

Debreu, G. (1959): *Theory of value: An axiomatic analysis of economic equilibrium*, vol. 17, Yale University Press.

Debreu, G. et al. (1954): "Representation of a preference ordering by a numerical function," *Decision Processes*, 3, 159–165.

Foley, D. K. (1966): *Resource Allocation and the Public Sector*, Yale University.

Lindner, C. and J. Rothe (2015): "Cake-cutting: Fair division of divisible goods," in *Economics and Computation: An Introduction to Algorithmic Game Theory, Computational Social Choice, and Fair Division*, Springer Nature, 393–491.

Mas-Colell, A., M. D. Whinston, J. R. Green, et al. (1995): *Microeconomic theory*, Oxford university press New York.

Procaccia, A. D. (2016): "Cake cutting algorithms," in *Chapter 13, Handbook of Computational Social Choice*, Cambridge University Press.

Sprumont, Y. (1991): "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica: Journal of the Econometric Society*, 509–519.

Stromquist, W. (2008): "Envy-free cake divisions cannot be found by finite protocols," *The Electronic Journal of Combinatorics*, 15, R11.

Thomson, W. (1994): "Resource-monotonic solutions to the problem of fair division when preferences are single-peaked," *Social Choice and Welfare*, 11, 205–223.

——— (2011): "Fair allocation rules," in *Handbook of Social Choice and Welfare*, Elsevier, vol. 2, 393–506.

——— (2016): "Introduction to the Theory of Fair Allocation," in *Chapter 11, Handbook of Computational Social Choice*, Cambridge University Press.

Thomson, W. and H. R. Varian (1984): "Theories of justice based on symmetry," *Discussion paper, Center for Research on Economic and Social Theory Research Seminar in Quantitative Economics.*

Varian, H. R. (1973): "Equity, envy, and efficiency," *Working papers, Department of Economics, Massachusetts Institute of Technology.*