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The Measurement of Welfare Change

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The measurement of welfare change*

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Abstract. We propose a class of measures of welfare change that are based on the generalized Gini social welfare functions. We analyze these measures in the context of a second-order dominance property that is akin to generalized Lorenz dominance as introduced by Shorrocks (1983) and Kakwani (1984). Because we consider welfare differences rather than welfare levels, the requisite equivalence result involves linear welfare functions (that is, those associated with the generalized Ginis) only, as opposed to the entire class of strictly increasing and strictly S-concave welfare indicators. Moving from second-order dominance to first-order dominance does not change this result significantly: for numerous pairs of income distributions, the generalized Ginis remain the only strictly increasing and strictly S-concave measures that are compatible with this first-order dominance condition phrased in terms of welfare change. Our final result provides a characterization of our measures of welfare change in the spirit of Weymark's (1981) original axiomatization of the generalized Gini welfare measures. *Journal of Economic Literature* Classification No.: D31.

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1 Introduction

Atkinson (1970), Kolm (1969) and Sen (1973) have provided the basic framework within which the modern approach to the measurement of inequality has developed. This approach explicitly endows measures of inequality with a normative interpretation. This is done by deriving inequality indices from social welfare functions defined on income distributions. Each index so derived inherits the distributional judgements embodied in the social welfare function from which it is derived. A reduction in inequality results in an increase in social welfare provided mean incomes remain unchanged.

Since the underlying social welfare functions are supposed to be *ordinal,* the derived measures of inequality are also ordinal. Hence, they can be used only to compare *levels* of inequality associated with the income distributions of, say, two countries, or of the same country at two points of time. However, it is also of considerable interest to ask questions such as "Has country A been more successful than country B in reducing inequality over the last decade?" A related but slightly different task is to compare the *change in social welfare* in country A to that of country B during a given period of time. Since the two countries may have a large difference in the rates of growth and since social welfare depends both on the size and the distribution of the cake, it may, in fact, be more appropriate to focus on comparisons of changes in social welfare. While this is the focus of this paper, we will also discuss briefly how to address the issue of comparing changes in the level of inequality.

A comparison of *changes* in levels of social welfare requires us to step out of the ordinal framework since the ranking of differences of welfare is not preserved under ordinal transformations of the welfare function. That is why we assume in this paper that the social welfare function has *cardinal* significance. This allows us to define a measure, V, of the change in social welfare between two income distributions, say x^0 and x^1. Furthermore, it is meaningful to compare *levels* of $V(x^0, x^1)$ and $V(y^0, y^1)$.

Having defined the measure of welfare change, V, we pursue two lines of inquiry. The first of these constitutes our main contribution. The generalized Gini welfare functions as introduced by Weymark (1981) constitute the entire class of *linear* functions of individual incomes satisfying strict increasingness and strict S-concavity (see Marshall and Olkin, 1979, for a detailed discussion of this property). Strict S-concavity is equivalent to the conjunction of anonymity and the well-established strict transfer principle that can be traced back to Pigou (1912) and Dalton (1920). The strict Pigou-Dalton transfer principle requires welfare (and hence equality) to go up when there is a transfer of income from a richer person to someone who is poorer without reversing their relative ranks in the distribution. Of course, the comparison of levels of welfare change can vary depending on *which* member of the class of generalized Ginis is used. Since there are no firm ethical reasons for preferring one generalized Gini function over another, it may not always be possible to arrive at unambiguous comparisons. So, we ask the question whether it is possible to define a *dominance* condition which, if satisfied, guarantees that the comparison of levels of welfare change give the same answer for *all* of the generalized Gini welfare functions. This question has, of course, been asked in the context of inequality of income distributions. Following Atkinson (1970) and Kolm (1969), Dasgupta, Sen and Starrett

(1973) proved the most general such result by showing that if two distributions have the same mean income, then the social welfare associated with income distribution x is higher than that of y according to any strictly S-concave welfare function if and only if the Lorenz curve of x lies everywhere above that of y. Thus, there is a precise dominance result for equality levels that corresponds to the class of strictly S-concave welfare functions. Shorrocks (1983) and Kakwani (1984) independently extended this result so as to be able to compare welfare levels of income distributions which do not have the same mean incomes. They scaled up the Lorenz curve of an income distribution by its mean income to obtain the *generalized Lorenz curve,* and showed that generalized Lorenz dominance of income distribution x over another distribution y also provides a necessary and sufficient condition for unambiguous welfare comparisons. That is, x has higher social welfare than y for *all* strictly increasing and strictly S-concave welfare functions if and only if its generalized Lorenz curve lies everywhere above that of y.

We too use generalized Lorenz curves in our analysis of comparisons of welfare change, but adapt them for our purpose. Consider any income distribution x^0 for a population of size n where individual incomes have been ranked in increasing order, so that $x_i \leq x_{i+1}$ for all i from 1 to $n-1$. Then, the generalized Lorenz curve of x is obtained by plotting the cumulative incomes of the lowest k income levels against each k for all k from 1 to n. Suppose now that there are two pairs of income distributions (all of population size n) indicating how income distributions have changed in countries A and B. Suppose also that all individual incomes have been arranged in increasing order. Then, we compare the sums of the cumulative *differences* between x^0 and x^1, and between y^0 and y^1. Put differently, we focus not on the generalized Lorenz curves themselves, but on the sums of the *vertical* differences between x^0 and x^1, and between y^0 and y^1. Our principal result is that the welfare change between x^0 and x^1 is at least as large as that between y^0 and y^1 for *all* generalized Gini differences if and only if the curve corresponding to the cumulative sums of differences between x^0 and x^1 lies everywhere above that of the corresponding curve for the y distributions.

Of course, this dominance is equivalent to second-order dominance of the difference between the x vectors and the y vectors. We then ask whether *first-order* dominance of the difference between the (ranked) x vectors over the difference between the (ranked) y vectors will imply unambiguous welfare change comparisons for a larger class of welfare functions. Clearly, this class would then contain the generalized Gini welfare functions since first-order dominance implies second-order dominance. However, we show the surprising result that even when there is first-order dominance in this sense, there are numerous pairs of distributions for which unambiguous welfare comparisons are possible only within the set of strictly increasing linear functions. So, once strict S-concavity is imposed, our result implies that first-order dominance does not buy very much that second-order dominance does not already give us. On the one hand, this demonstrates that unambiguous comparisons of welfare change are very hard to make. At the same time, however, this observation can be used as a forceful argument in favor of the generalized Gini welfare functions and their associated measures of welfare change.

As a secondary task, we specify some appealing axioms or properties of V and characterize the class of welfare change measures that are based on the generalized Gini welfare functions. Our proof closely follows that of Weymark (1981), the principal difference being that our axioms are imposed on the measure of welfare change rather than on the welfare function. In addition, our axioms are also slightly different allowing for a shorter proof.

2 Measures of welfare change

Suppose that there are $n \geq 2$ individuals in a society. A measure of welfare change is a function $V\colon \mathbb{R}^{2n}_{+} \to \mathbb{R}$ and we interpret $V(x^0, x^1)$ as an indicator of the improvement (or deterioration) associated with moving from last period's income distribution $x^0 \in \mathbb{R}^n_+$ to the current distribution $x^1 \in \mathbb{R}^n_+$. Our objective is to find a class of functions V that can be expressed in terms of a difference in welfare levels. That is, we require V to possess the following property.

Welfare difference compatibility. There exists a function $W\colon \mathbb{R}^n_+ \to \mathbb{R}$ such that, for all $x^0, x^1 \in \mathbb{R}^n_+$,

$$V(x^0, x^1) = W(x^1) - W(x^0). \tag{1}$$

We assume throughout that V satisfies a plausible monotonicity property.

Strict monotonicity. V is strictly increasing in x^1.

It follows immediately that if V is strictly monotonic and a function W as in (1) exists, this function W must be strictly increasing. Moreover, using (1) and the strict increasingness of W, it follows that V is strictly decreasing in x^0.

Anonymity requires that the individuals in a society be treated impartially, paying no attention to their identities. The strict transfer principle is an essential equity requirement that ensures welfare (and equality) to increase as a consequence of a rank-preserving progressive transfer. As is well-known, the conjunction of anonymity and the strict transfer principle is equivalent to strict S-concavity. To introduce this property formally, we require another definition. An $n \times n$ matrix D is doubly stochastic if its entries are non-negative and all row sums and column sums are equal to one. We require V to be strictly S-concave in its second argument.

Strict S-concavity in the second argument. For all $x^0, x^1 \in \mathbb{R}^n_+$ and for all doubly stochastic $n \times n$ matrices D, $V(x^0, Dx^1) \geq V(x^0, x^1)$ and, if Dx^1 is not a permutation of x^1, this inequality is strict.

Clearly, if a function W as in (1) exists, it must be strictly S-concave if V is strictly S-concave in its second argument. Moreover, the conjunction of welfare difference compatibility and the strict S-concavity of V in its second argument implies that V is strictly S-convex in its first argument (where strict S-convexity is obtained if the inequality in the definition of strict S-concavity is reversed).

The set of bottom-first-ordered permutations of the elements of $\mathbb{R}^n_+$ is given by

$$B = \{x \in \mathbb{R}^n_+ \mid x_1 \leq \ldots \leq x_n\}.$$

Let

$$A = \{\alpha \in \mathbb{R}^n_{++} \mid \alpha_1 > \ldots > \alpha_n\}.$$

A welfare function W is a generalized Gini welfare function if there exists a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$ such that, for all $x \in \mathbb{R}^n_+$,

$$W(x) = \sum_{i=1}^{n} \alpha_i \tilde{x}_i \tag{2}$$

where $\tilde{x} \in B$ is a bottom-first-ordered permutation of x. Thus, the weights α_i are assigned to the positions in an income distribution, where higher incomes receive lower weights in order to ensure that the resulting welfare function respects the strict transfer principle. The corresponding generalized Gini measure of welfare change is given by

$$V(x^0, x^1) = \sum_{i=1}^{n} \alpha_i \tilde{x}^1_i - \sum_{i=1}^{n} \alpha_i \tilde{x}^0_i \tag{3}$$

for all $(x^0, x^1) \in \mathbb{R}^{2n}_+$. The measure of welfare change associated with the parameter vector $\alpha \in A$ is denoted by V_α. The class of all generalized Gini measures of welfare change is given by

$$\mathcal{V}_G = \{V_\alpha \mid \alpha \in A\}.$$

Because we restrict attention to anonymous measures of welfare and welfare change, it involves no loss of generality to assume that x is bottom-first ordered.

3 Dominance properties

To simplify notation, we define

$$\Delta x_i = x^1_i - x^0_i \;\text{ and }\; \Delta y_i = y^1_i - y^0_i \;\text{ for all } i \in \{1, \ldots, n\}$$

for all bottom-first-ordered income distributions $x^0, x^1, y^0, y^1 \in B$. The following dominance property is a welfare-change adaptation of the generalized Lorenz criterion; see Shorrocks (1983).

Second-order dominance. For all $x^0, x^1, y^0, y^1 \in B$, (x^0, x^1) second-order dominates (y^0, y^1) if and only if

$$\sum_{i=1}^{k} (\Delta x_i - \Delta y_i) \geq 0 \;\text{ for all } k \in \{1, \ldots, n\}.$$

Our objective is to derive a condition on any two pairs of distributions (x^0, x^1) and (y^0, y^1) that will enable us to state that the welfare change between (x^0, x^1) is greater or smaller than the welfare change between (y^0, y^1) for *all* measures of welfare change in the class $\mathcal{V}_G$. So, we want to rule out cases where there are two parameter vectors $\alpha, \alpha' \in A$ such that

$$V_\alpha(x^0, x^1) \geq V_\alpha(y^0, y^1) \;\text{ and }\; V_{\alpha'}(x^0, x^1) < V_{\alpha'}(y^0, y^1).$$

The following lemma provides a condition that prevents such reversals.

Lemma 1. *For all* $x^0, x^1, y^0, y^1 \in B$,

$$V(x^0, x^1) \geq V(y^0, y^1) \;\; \textit{for all } V \in \mathcal{V}_G$$

if and only if

$$\sum_{i=1}^{k} \alpha_i(\Delta x_i - \Delta y_i) \geq 0 \;\; \textit{for all } k \in \{1, \ldots n\} \textit{ and for all } \alpha \in A. \tag{4}$$

Proof. Sufficiency of (4) follows from the definition of the elements of $\mathcal{V}_G$.

To prove necessity, let $V_\alpha(x^0, x^1) \geq V_\alpha(y^0, y^1)$ for some $\alpha \in A$. Therefore, by definition

$$\sum_{i=1}^{n} \alpha_i(\Delta x_i - \Delta y_i) \geq 0.$$

Suppose there exists a $k < n$ such that

$$\sum_{i=1}^{k} \alpha_i(\Delta x_i - \Delta y_i) < 0.$$

Let $c > 1$ and define a vector $\alpha^c \in A$ as follows. For all $i \in \{1, \ldots, n\}$,

$$\alpha_i^c = \begin{cases} \alpha_i & \text{if } i \leq k, \\ \frac{1}{c}\alpha_i & \text{if } i > k. \end{cases}$$

Clearly, $\alpha^c \in A$ for all $c > 1$, and there exists c^* sufficiently large such that

$$\sum_{i=1}^{n} \alpha_i^{c^*}(\Delta x_i - \Delta y_i) < 0,$$

that is, $V_{\alpha^{c^*}}(y^0, y^1) > V_{\alpha^{c^*}}(x^0, x^1)$. This shows that if (4) is not satisfied, then there are two measures of welfare change in $\mathcal{V}_G$ which differ in their ranking of the pairs (x^0, x^1) and (y^0, y^1), a contradiction that completes the proof. ■

We prove one more lemma before stating our main result.

Lemma 2. *Let* $x^0, x^1, y^0, y^1 \in B$ *and suppose that* (x^0, x^1) *second-order dominates* (y^0, y^1). *Then, for all* $\alpha \in A$ *and for all* $k \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{k} \alpha_i(\Delta x_i - \Delta y_i) \geq \alpha_k \sum_{i=1}^{k}(\Delta x_i - \Delta y_i).$$

Proof. Clearly, the claim is true for $k = 1$. By way of induction, suppose that it is true for all $k' < k$. Then it follows that

$$\alpha_k(\Delta x_k - \Delta y_k) + \sum_{i=1}^{k-1} \alpha_i(\Delta x_i - \Delta y_i) \geq \alpha_k(\Delta x_k - \Delta y_k) + \alpha_{k-1} \sum_{i=1}^{k-1}(\Delta x_i - \Delta y_i).$$

Hence,

$$\sum_{i=1}^{k} \alpha_i(\Delta x_i - \Delta y_i) \geq \alpha_k \sum_{i=1}^{k} (\Delta x_i - \Delta y_i)$$

since $\alpha_{k-1} > \alpha_k$ because $\alpha \in A$ and $\sum_{i=1}^{k-1}(\Delta x_i - \Delta y_i) \geq 0$ because (x^0, x^1) second-order dominates (y^0, y^1). ■

We can now state the main result of this section.

Theorem 1. *For all* $x^0, x^1, y^0, y^1 \in B$,

$$V(x^0, x^1) \geq V(y^0, y^1) \;\; \text{for all } V \in \mathcal{V}_G$$

if and only if (x^0, x^1) *second-order dominates* (y^0, y^1).

Proof. Suppose first that (x^0, x^1) second-order dominates (y^0, y^1). Then, by definition,

$$\sum_{i=1}^{k} (\Delta x_i - \Delta y_i) \geq 0 \;\; \text{for all } k \in \{1, \ldots, n\}. \tag{5}$$

Take any $\alpha \in A$. Then the inequality

$$\alpha_1(\Delta x_1 - \Delta y_1) \geq 0$$

follows from setting $k = 1$ in (5) and the fact that $\alpha_1 > 0$. Suppose that, for some $K \in \{2, \ldots, n\}$,

$$\sum_{i=1}^{k} \alpha_i(\Delta x_i - \Delta y_i) \geq 0 \;\; \text{for all } k \in \{1, \ldots, K-1\}.$$

We want to show that

$$\sum_{i=1}^{K} \alpha_i(\Delta x_i - \Delta y_i) \geq 0.$$

By (5) and Lemma 2,

$$\sum_{i=1}^{K} \alpha_i(\Delta x_i - \Delta y_i) \geq \alpha_K \sum_{i=1}^{k} (\Delta x_i - \Delta y_i) \geq 0.$$

Since $\alpha \in A$ was chosen arbitrarily, this inequality together with Lemma 1 establishes that $V(x^0, x^1) \geq V(y^0, y^1)$ for all $V \in \mathcal{V}_G$.

Now suppose that $V(x^0, x^1) \geq V(y^0, y^1)$ for all $V \in \mathcal{V}_G$. We need to show that (x^0, x^1) second-order dominates (y^0, y^1). In view of Lemma 1, it is sufficient to prove that if (4) is satisfied, then (x^0, x^1) second-order dominates (y^0, y^1).

Pick any $\alpha \in A$. Then,

$$\sum_{i=1}^{k} \alpha_i(\Delta x_i - \Delta y_i) \geq 0 \text{ for all } k \in \{1, \ldots, n\}.$$

Clearly, $\Delta x_1 \geq \Delta y_1$ since $\alpha_1 > 0$. Let $K \in \{2, \ldots, n\}$ and assume that

$$\sum_{i=1}^{k} (\Delta x_i - \Delta y_i) \geq 0 \text{ for all } k \in \{1, \ldots, K-1\}.$$

Suppose that

$$\sum_{i=1}^{K} (\Delta x_i - \Delta y_i) < 0.$$

Multiplying by $\alpha_K > 0$, we obtain

$$\sum_{i=1}^{K} \alpha_K(\Delta x_i - \Delta y_i) < 0. \tag{6}$$

Let $\varepsilon \in \mathbb{R}_{++}$ and define $\alpha^\varepsilon \in A$ as follows. For all $i \in \{1, \ldots, n\}$,

$$\alpha_i^\varepsilon = \begin{cases} \alpha_K + (K - i)\varepsilon & \text{if } i < K, \\ \alpha_i & \text{if } i \geq K. \end{cases}$$

From (4), we know that

$$\sum_{i=1}^{K} \alpha_i^\varepsilon(\Delta x_i - \Delta y_i) \geq 0 \text{ for all } \varepsilon \in \mathbb{R}_{++}.$$

But (6) implies

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{K} \alpha_i^\varepsilon(\Delta x_i - \Delta y_i) = \alpha_K \sum_{i=1}^{K} (\Delta x_i - \Delta y_i) < 0,$$

a contradiction to (4) that completes the proof of the theorem. ■

A comparison with the result obtained by Shorrocks (1983) illustrates the impact of considering welfare differences rather than welfare levels. Because of the linearity inherent in difference comparisons, not all strictly increasing and strictly S-concave welfare functions have to agree in order to obtain an equivalence result but only those among them that are linear—that is, those corresponding to the generalized Ginis.

Thus, as established in the above theorem, welfare functions other than the generalized Ginis cannot be employed in an equivalence result that involves our second-order dominance condition. This raises the question of whether a more stringent dominance definition can accommodate a more general class of functions. For instance, we may want to impose a first-order dominance property for welfare differences, defined as follows. Again, we assume that, without loss of generality, the income distributions x^0, x^1, y^0, y^1 are in B, and Δx_i and Δy_i are defined as above.

First-order dominance. For all $x^0, x^1, y^0, y^1 \in B$, (x^0, x^1) first-order dominates (y^0, y^1) if and only if

$$\Delta x_i - \Delta y_i \geq 0 \text{ for all } i \in \{1, \ldots, n\}.$$

Because first-order dominance implies second-order dominance, the generalized Gini measures of welfare change are compatible with this dominance property. There may be additional measures that can be accommodated in the first-order case but, as illustrated below, all of them must be based on linear measures as well.

Let $\mathcal{F}$ be a class of measures of welfare change. If the members of $\mathcal{F}$ are to be compatible with the first-order dominance criterion, it must be the case that if (x^0, x^1) first-order dominates (y^0, y^1), then

$$V(x^0, x^1) \geq V(y^0, y^1) \text{ for all } V \in \mathcal{F}$$

or, in terms of the underlying welfare functions W,

$$W(x^1) - W(x^0) \geq W(y^1) - W(y^0). \tag{7}$$

Consider $x^1, y^0 \in B$ and let $x^0 = y^1 = (x^1 + y^0)/2$, that is, the distributions x^0 and y^1 are both equal to the arithmetic mean of x^1 and y^0. Therefore, by definition,

$$\Delta x_i - \Delta y_i = (x_i^1 - x_i^0) - (y_i^1 - y_i^0) = 0 \text{ for all } i \in \{1, \ldots, n\}.$$

Because $x^0 = y^1 = (x^1 + y^0)/2$, (7) requires that

$$\begin{aligned} W(x^1) - W(x^0) = W(y^1) - W(y^0) \;&\Leftrightarrow\; W(x^1) - W(x^0) = W(x^0) - W(y^0) \\ &\Leftrightarrow\; 2W(x^0) = W(x^1) + W(y^0) \\ &\Leftrightarrow\; W\left(\frac{1}{2}x^1 + \frac{1}{2}y^0\right) = \frac{1}{2}W(x^1) + \frac{1}{2}W(y^0), \end{aligned}$$

a condition that, along with strict increasingness, requires W to be a strictly increasing affine function within the bottom-first-ordered subspace of $\mathbb{R}_+^n$. This, in turn, means that the associated measure of welfare change is linear. Because the only increasing functions with that property other than the generalized Ginis are such that the parameter vectors α do not respect the inequalities that define membership in A, it follows that these additional functions fail to satisfy strict S-concavity. Thus, for numerous pairs of income distributions, even this first-order dominance condition does not allow for measures other than the generalized Ginis if this fundamental equity property is to be retained.

We conclude this section with an observation that follows from Theorem 1. Consider the question of making similar unambiguous comparisons of changes in the level of inequality, where the measure of inequality is derived from a social welfare function according to the

Atkinson-Kolm-Sen approach alluded to in the introduction. Take two pairs of income distributions (x^0, x^1) and (y^0, y^1) each with the same (positive) mean income. Let

$$Z(x^0, x^1) = I(x^1) - I(x^0)$$

be a measure of inequality change, analogous to V. Suppose, moreover, that

$$I(x) = 1 - \frac{x_e}{\mu(x)},$$

where $\mu(x)$ is mean income and x_e is the *equally-distributed-equivalent income* corresponding to the income distribution x and the welfare function W. That is, x_e is implicitly defined by

$$W(x_e, \ldots, x_e) = W(x).$$

The index I has an intuitive normative interpretation: it measures the percentage shortfall of the equally-distributed-equivalent income from average income, where this shortfall is attributable to the presence of inequality in the income distribution under consideration.

Let $\mathcal{I}_G$ be the class of inequality measures that are derived from the class of generalized Gini welfare functions, and let $\mathcal{Z}_G$ be the set of measures of inequality change that represent the difference of inequality levels where the inequality index is some member of $\mathcal{I}_G$. In view of Theorem 1, the following result is immediate.

Theorem 2. *For all $x^0, x^1, y^0, y^1 \in B$ with the same mean income,*

$$Z(x^0, x^1) \leq Z(y^0, y^1) \;\; \text{for all } Z \in \mathcal{Z}_G$$

if and only if (x^0, x^1) second-order dominates (y^0, y^1).

Note that since we restrict the four distributions to have the same mean income, we could also state the above theorem in terms of second-order dominance of the vertical differences in the *Lorenz curves.*

4 A characterization

We conclude this paper by providing a characterization of the generalized Gini measures of welfare change. There clearly is a strong resemblance to Weymark's (1981) axiomatization but some arguments in his proof can be simplified here because our list of axioms is slightly different from his.

In addition to welfare difference compatibility, strict monotonicity and strict S-concavity in the second argument, we use the following two properties that are well-established in the context of welfare functions. They continue to have strong intuitive appeal when formulated for a measure of welfare change.

Positive linear homogeneity is a standard requirement for welfare functions that can be expressed analogously as a property of a measure of welfare change.

Positive linear homogeneity. For all $x^0, x^1 \in \mathbb{R}^n_+$ and for all $\lambda \in \mathbb{R}_{++}$,

$$V(\lambda x^0, \lambda x^1) = \lambda V(x^0, x^1).$$

Our final axiom is an independence condition that is restricted to income distributions in which all incomes are ranked from lowest to highest. Recall that B is the set of bottom-first-ordered permutations of the elements of $\mathbb{R}^n_+$. The welfare function analogue of following property appears in Weymark (1981, p. 418).

Weak independence of income source. For all $x^0, x^1, y^0, y^1, z \in B$,

$$V(x^0 + z, x^1 + z) \geq V(y^0 + z, y^1 + z) \Leftrightarrow V(x^0, x^1) \geq V(y^0, y^1). \tag{8}$$

This axiom implies that W has the corresponding property as defined in Weymark (1981). Using (1), it follows that (8) is equivalent to

$$W(x^1 + z) - W(x^0 + z) \geq W(y^1 + z) - W(y^0 + z) \Leftrightarrow W(x^1) - W(x^0) \geq W(y^1) - W(y^0)$$

for all $x^0, x^1, y^0, y^1, z \in B$. Setting $x^0 = y^0$, this simplifies to

$$W(x^1 + z) \geq W(y^1 + z) \Leftrightarrow W(x^1) \geq W(y^1)$$

for all $x^1, y^1, z \in B$, which is Weymark's (1981) condition.

We can now state the result of this section.

Theorem 3. *A measure of welfare change V satisfies welfare difference compatibility, strict monotonicity, strict S-concavity in the second argument, positive linear homogeneity and weak independence of income source if and only if V is a generalized Gini measure of welfare change with a corresponding generalized Gini welfare function W.*

Proof. That the generalized Gini measures of welfare change satisfy the axioms of the theorem statement is straightforward to verify.

Conversely, suppose that V is a measure of welfare change satisfying the axioms. By anonymity (which follows from strict S-concavity in the second argument), it is sufficient to show that (2) and (3) are true for bottom-first-ordered permutations of the requisite income distributions. As mentioned in the text, the welfare function W (which exists as a consequence of welfare difference compatibility) inherits the properties of strict increasingness, strict S-concavity and weak independence of income source suitably formulated for welfare functions.

We now show that the restriction of W to bottom-first-ordered permutations must be an increasing transformation of a strictly increasing linear function. Because we assume that V satisfies positive linear homogeneity, the argument used in the proof of Weymark's (1981) Theorem 3 can be simplified. To do so, we first prove that the restriction of any level set of W to B is a convex set. Let $z, z' \in B$ be in the same level set of W so that $W(z) = W(z')$. Using (1) and the positive linear homogeneity of V, it follows that

$$W(z) = W(z') \Leftrightarrow V(z, z') = 0 \Leftrightarrow V(\lambda z, \lambda z') = 0 \Leftrightarrow W(\lambda z) = W(\lambda z')$$

for all $\lambda \in \mathbb{R}_{++}$. Letting $\theta \in (0,1)$, it follows that

$$W(z) = W(z') \;\Leftrightarrow\; W((1-\theta)z) = W((1-\theta)z'). \tag{9}$$

Adding θz to both $(1-\theta)z$ and $(1-\theta)z'$, weak independence of income source implies that

$$W((1-\theta)z) = W((1-\theta)z') \;\Leftrightarrow\; W(\theta z + (1-\theta)z) = W(z) = W(\theta z + (1-\theta)z')$$

and, combined with (9), we obtain

$$W(z) = W(z') \;\Leftrightarrow\; W(z) = W(\theta z + (1-\theta)z')$$

for all $z, z' \in B$ in the same level set of W and for all $\theta \in (0,1)$, which implies that the requisite level set is convex. Because W is strictly increasing, it follows that the restriction of W to B is an increasing transformation of a strictly increasing linear function. Thus, there exist $\beta_1, \ldots, \beta_n \in \mathbb{R}_{++}$ and an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ such that, for all $x \in B$,

$$W(x) = \phi\left(\sum_{i=1}^{n} \beta_i x_i\right). \tag{10}$$

By strict S-concavity and because the elements of B are bottom-first-ordered, it follows that $\beta_1 > \ldots > \beta_n$. By anonymity,

$$W(x) = \phi\left(\sum_{i=1}^{n} \beta_i \tilde{x}_i\right)$$

for all $x \in \mathbb{R}^n_+$.

Using (10) and noting that, for any $p, q \in \mathbb{R}_+$, two distributions x and y can be chosen so that $\sum_{i=1}^{n} \beta_i \tilde{x}^0_i = p$ and $\sum_{i=1}^{n} \beta_i \tilde{x}^1_i = q$, positive linear homogeneity requires that

$$\phi(\lambda q) - \phi(\lambda p) = \lambda(\phi(q) - \phi(p)) \tag{11}$$

for all $p, q \in \mathbb{R}_+$ and for all $\lambda \in \mathbb{R}_{++}$. Setting $p > 0$, $q = 0$ and $\lambda = 1/p$ in (11), it follows that

$$\phi(0) - \phi(1) = (\phi(0) - \phi(p))/p$$

and, solving for $\phi(p)$, we obtain

$$\phi(p) = (\phi(1) - \phi(0))p + \phi(0) = \Delta y p + \delta$$

where $\Delta y = \phi(1) - \phi(0)$ is positive because ϕ is increasing and $\delta = \phi(0)$ is a real number. Therefore, ϕ is an increasing affine function and, setting $\alpha_i = \Delta y \beta_i$ for all $i \in \{1, \ldots, n\}$, it follows that $\alpha_1 > \ldots > \alpha_n$ and

$$W(x) = \sum_{i=1}^{n} \alpha_i \tilde{x}_i$$

for all $x \in \mathbb{R}^n_+$. Using welfare difference compatibility, we obtain

$$V(x^0, x^1) = \sum_{i=1}^{n} \alpha_i \tilde{x}^1_i - \sum_{i=1}^{n} \alpha_i \tilde{x}^0_i$$

for all $(x^0, x^1) \in \mathbb{R}^{2n}_+$. ■

5 Concluding remarks

In order to establish a dominance criterion that allows for welfare changes to be compared across societies with different population sizes, one possible approach consists of replicating the requisite societies and employing the dominance criterion that corresponds to the larger population. Specifically, if we have pairs of distributions $(x^0, x^1) \in \mathbb{R}_+^{2n}$ and $(y^0, y^1) \in \mathbb{R}_+^{2m}$ where $n \neq m$, we can consider an m-fold replication of (x^0, x^1) and an n-fold replication of (y^0, y^1) and apply the dominance criterion for population size nm to the replicated distributions. Of course, implicit in such a procedure—which is also suggested by Shorrocks (1983)—is some suitable notion of a principle of population, ensuring that such replications do not distort welfare-change-relevant features of the original distributions. This observation leads us to the *single-parameter Ginis,* which are characterized by Donaldson and Weymark (1980) by means of the principle of population. A similar variable-population result that employs a recursivity property characterizes the *single-series Ginis;* see Bossert (1990). In analogy to our characterization that parallels Weymark's (1981) axiomatization, these variable-population extensions can be adjusted to our setting so as to apply to measures of welfare change.

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